

Perturbations and Projections of Kalman-Bucy Semigroups Motivated by Methods in Data Assimilation

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Abstract

The purpose of this work is to analyse the effect of various perturbations and projections of Kalman-Bucy semigroups and Riccati equations. The original motivation was to understand the behaviour of various regulation methods used in ensemble Kalman filtering (**EnKF**). For example, covariance inflation-type methods (perturbations) and covariance localisation methods (projections) are commonly used in the **EnKF** literature to ensure well-posedness of the sample covariance (e.g. sufficient rank) and to ‘move’ the sample covariance closer (in some sense) to the Riccati flow of the true Kalman filter. In the limit, as the number of samples tends to infinity, these methods drive the sample covariance toward a solution of a perturbed, or projected, version of the standard (Kalman-Bucy) differential Riccati equation. The behaviour of this modified Riccati equation is investigated here. Results concerning continuity (in terms of the perturbations), boundedness, and convergence of the Riccati flow to a limit are given. In terms of the limiting filters, results characterising the error between the perturbed/projected and nominal conditional distributions are given. New projection-type models and ideas are also discussed within the **EnKF** framework; e.g. projections onto so-called Bose-Mesner algebras. This work is generally important in understanding the limiting bias in both the **EnKF** empirical mean and covariance when applying regularisation. Finally, we note the perturbation and projection models considered herein are also of interest on their own, and in other applications such as differential games, control of stochastic and jump processes, and robust control theory, etc.

1 Introduction

The purpose of this work is to analyse a number of perturbations and projections of Kalman-Bucy [1, 2] semigroups and of the associated (matrix differential) Riccati flow.

The prime motivating application for this work is the ensemble Kalman filter (**EnKF**) [19] and the various ‘regularisation’ methods used to ensure well-posedness of the sample covariance (e.g. sufficient rank) and to ‘move’ the sample covariance closer (in some sense) to the Riccati flow of the true Kalman filter [1, 2]. For example, two common forms of regularisation are covariance inflation-type methods (perturbations) and so-called covariance localisation methods (projections). Covariance inflation is a simple idea that involves adding some positive-definite matrix to the sample covariance in order to increase its rank [14]; i.e. more specifically to account for an underrepresentation of the true variance due to a potentially inferior sample size. Separately, the idea of covariance localization involves multiplying (element-wise) the **EnKF** sample covariance matrix via Schur (or Hadamard) products with certain sparse ‘masking’ matrices with the intent of reducing spurious long-range correlations and increasing the sample covariance rank [23]. See [20] for an empirical

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examination of both types of regularisation. In these two cases, choosing the right inflation or localization is non-trivial and numerous ideas exist; e.g. [16, 25, 17, 27, 18]. Other related, and/or more subtle, regularisation methods exist and we will cover more general models in more detail in later sections; see also [21, 15, 28, 24, 26, 22] for related **EnKF** methodology.

Note that the total literature on **EnKF** methodology is too broad to cover adequately here. Results on **EnKF** convergence are recent (relative to this work) and concern, e.g., weak convergence with sample size [29, 30], and stability [31, 32, 33, 34, 35], etc. The articles [33, 36] concern stability and robustness of the **EnKF** in the presence of specific inflation and localisation methods.

From a purely mathematical vantage, regularisation amounts to studying various projections and perturbations of the ‘standard’ Riccati flow (viz [1, 2]). The analytical behaviour of general projections and perturbations are a major focus of this study. New ideas concerning projections relevant to the **EnKF** are also introduced. Given this analysis, we then study the (nonlinear) Kalman-Bucy diffusion [2] and provide a number of concentration/contraction-type convergence results between the corresponding perturbed/projected diffusion and the optimal Kalman-Bucy diffusion. We study convergence in the mean-square sense and also in terms of the limiting law of the diffusion.

While methods in data assimilation and ensemble Kalman filtering are the main drivers of this work, the types of perturbations considered herein are more widely relevant: For example, our analysis captures those perturbations of the ‘standard’ Riccati flow that arise in, e.g., linear quadratic differential games [6, 12, 8], in the control of linear stochastic jump systems [9, 4], in certain robust and H^∞ control settings [10, 5], etc; see also the early work of Wonham [13] in linear-quadratic stochastic control. We also highlight the text [3, e.g. Chap. 6] and the references therein. Separately, a specific projected Riccati flow is studied in [7]. Going forward, we primarily rely on **EnKF** motivators, but we emphasise that the mathematical development is more broadly applicable.

Further introduction, discussion, and background is given in later subsections with a more technical focus. The organisation of this article is as follows:

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1.1 Kalman-Bucy diffusions

The notation used throughout this article is introduced later in Section 1.4. However, the set-up in this section is relatively standard. Consider a time homogeneous linear-Gaussian filtering model of the following form

$$\begin{cases} dX_t &= A X_t dt + R^{1/2} dW_t \\ dY_t &= C X_t dt + \Sigma^{1/2} dV_t \end{cases} \quad (1)$$

where (W_t, V_t) is an $(r+r')$ -dimensional Brownian motion, X_0 is a r -valued Gaussian random vector (independent of (W_t, V_t)) with mean $\mathbb{E}(X_0)$ and covariance matrix P_0 , the symmetric matrices $R^{1/2}$ and $\Sigma^{1/2}$ are invertible, A is a square $(r \times r)$ -matrix, C is an $(r' \times r)$ -matrix, and $Y_0 = 0$. We let $\mathcal{F}_t = \sigma(Y_s, s \leq t)$ be the filtration generated by the observation process.

It is well-known that the conditional distribution η_t of the signal state X_t given \mathcal{F}_t is a r -dimensional Gaussian distribution with a mean and covariance matrix given by

$$\hat{X}_t := \mathbb{E}(X_t | \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}((X_t - \mathbb{E}(X_t | \mathcal{F}_t))(X_t - \mathbb{E}(X_t | \mathcal{F}_t))')$$

given by the Kalman-Bucy and the Riccati equations

$$d\hat{X}_t = A \hat{X}_t dt + P_t C' \Sigma^{-1} (dY_t - C \hat{X}_t dt) \quad \text{with} \quad \partial_t P_t = \text{Ricc}(P_t). \quad (2)$$

In the above display, Ricc stands for the Riccati drift function from \mathbb{S}_r^+ into \mathbb{S}_r defined for any $Q \in \mathbb{S}_r^+$ by

$$\text{Ricc}(Q) = AQ + QA' - QSQ + R \quad \text{with} \quad S := C' \Sigma^{-1} C \quad (3)$$

We now consider the conditional nonlinear McKean-Vlasov type diffusion process

$$d\bar{X}_t = A \bar{X}_t dt + R^{1/2} d\bar{W}_t + \mathcal{P}_{\eta_t} C' \Sigma^{-1} [dY_t - (C \bar{X}_t dt + \Sigma^{1/2} d\bar{V}_t)] \quad (4)$$

where $(\bar{W}_t, \bar{V}_t, \bar{X}_0)$ are independent copies of (W_t, V_t, X_0) (thus independent of the signal and the observation path). In the above displayed formula \mathcal{P}_{η_t} stands for the covariance matrix

$$\mathcal{P}_{\eta_t} = \eta_t [(e - \eta_t(e))(e - \eta_t(e))'] \quad \text{with} \quad \eta_t := \text{Law}(\bar{X}_t | \mathcal{F}_t) \quad \text{and} \quad e(x) := x. \quad (5)$$

We shall call this probabilistic model the Kalman-Bucy (nonlinear) diffusion process.

The ensemble Kalman-Bucy filter (**EnKF**) coincides with the mean-field particle approximation of the nonlinear diffusion process (4). To be more precise we let $(\bar{W}_t^i, \bar{V}_t^i, \xi_0^i)_{1 \leq i \leq N}$ be N independent copies of $(\bar{W}_t, \bar{V}_t, \bar{X}_0)$. In this notation, the **EnKF** is given by the McKean-Vlasov type interacting diffusion process

$$\begin{cases} d\xi_t^i &= A \xi_t^i dt + R^{1/2} d\bar{W}_t^i + p_t C' \Sigma^{-1} [dY_t - (C \xi_t^i dt + \Sigma^{1/2} d\bar{V}_t^i)] \\ i &= 1, \dots, N \end{cases} \quad (6)$$

with the rescaled particle covariance matrices $p_t := (1 - N^{-1})^{-1} \mathcal{P}_{\eta_t^N}$ defined in terms of the empirical measures $\eta_t^N := N^{-1} \sum_{1 \leq i \leq N} \delta_{\xi_t^i}$.

1.2 Perturbations and projections

From a pure mathematical position, our model of perturbation or projection is motivated by methodology that replaces the sample covariance p_t in (6) by some matrix $\pi(p_t)$, where $\pi : \mathbb{S}_r^+ \mapsto \mathbb{S}_r^+$ is some judiciously chosen mapping. These methods coincide with the mean field particle approximation of the nonlinear diffusion \overline{X}_t^π defined by (4) with \mathcal{P}_{η_t} replaced by $\pi(\mathcal{P}_{\eta_t^\pi})$, i.e.,

$$d\overline{X}_t^\pi = A \overline{X}_t^\pi dt + R^{1/2} d\overline{W}_t + \pi(\mathcal{P}_{\eta_t^\pi}) C' \Sigma^{-1} \left[dY_t - \left(C \overline{X}_t^\pi dt + \Sigma^{1/2} d\overline{V}_t \right) \right] \quad (7)$$

where $\eta_t^\pi = \text{Law}(\overline{X}_t^\pi \mid \mathcal{F}_t)$. The initial state \overline{X}_0^π is some Gaussian random variable with some covariance matrix $\mathcal{P}_{\eta_0^\pi}$. We expect the empirical average of the EnKF system associated with (7) to converge to the Kalman-Bucy filter defined by (2) except with P_t replaced by the matrix $\pi(P_t)$. From the statistical viewpoint, the Kalman-Bucy filter $\hat{X}^\pi := \mathbb{E}(\overline{X}_t^\pi \mid \mathcal{F}_t)$ associated with (7) captures the limiting bias of the EnKF empirical mean, introduced by a given perturbation or projection. The nonlinear diffusion (7) is well posed and the flow of covariance matrices $P_t^\pi = \mathcal{P}_{\eta_t^\pi}$ satisfies

$$\begin{aligned} \partial_t P_t^\pi &= \text{Ricc}^\pi(P_t^\pi) \\ &:= [A - \pi(P_t^\pi)S] P_t^\pi + P_t^\pi [A - \pi(P_t^\pi)S]' + R + \pi(P_t^\pi)S\pi(P_t^\pi) \end{aligned} \quad (8)$$

as soon as π is chosen so that (8) has a unique positive definite solution. A proof of this assertion is provided in the appendix; see page 41. This equation captures the covariance flow of the limiting perturbed/projected Kalman-Bucy filter $\hat{X}^\pi := \mathbb{E}(\overline{X}_t^\pi \mid \mathcal{F}_t)$ associated with (7) and consequently it captures the bias in the limiting EnKF sample covariance as $N \rightarrow \infty$. In general, the bias in this covariance flow (compared with (2)) is the root cause of the limiting bias in the (regularised) EnKF empirical mean (not vice-versa). Hence, *it is this perturbed or projected Riccati equation (8) that is the main object of study in this work.*

In the further development we shall distinguish and analyze the two different cases:

$$1) \quad \pi = id + \Delta \quad \text{with} \quad \Delta \simeq 0 \quad \text{or} \quad 2) \quad \pi \circ \pi = \pi \quad (9)$$

where id stands for the identity mapping.

The first class of model can be thought of as a local perturbation mapping. These mappings are associated to some parameter that describe the level of perturbation. This model includes the variance inflation techniques discussed in Section 4.1, mean-repulsion type perturbations discussed in Section 4.2, and Stein-Shrinkage models presented in Section 4.5, among others.

The second class of model corresponds to projection-type mappings such as masked projections (or localization methods) discussed in Section 4.3 and projection mappings on Bose-Mesner algebras discussed in Section 4.4.

We also show later that the first class of model can capture most projection-type perturbations, or more general classes of test-type driving estimators; see the broader discussion in Section 4.

1.2.1 Some relevant commentary

A central feature of the EnKF is the sample-based covariance estimation of the solution to the differential Riccati equation using a collection of interacting Kalman-Bucy filtering samples. In contrast to conventional covariance estimates based on independent random samples, the EnKF is based on interacting samples. These samples are sequentially updated by some noisy observation process through a gain matrix that itself depends on the sample covariance matrices. The corresponding process is highly nonlinear (even when the true signal and observation model is linear). In high dimensions, the interacting particle estimation of the Riccati solution experiences the same difficulties as any conventional sample covariance estimator. For example:

- The sample covariance p_t is the sample mean of N unit-rank matrices and by the rank-nullity theorem has null eigenvalues when $N < r$. Thus, in some principal directions, the **EnKF** is driven solely by the signal diffusion. With unstable signal drift matrices, the **EnKF** will exhibit divergence as it is not corrected by the innovation process. In this setting, one cannot design a stable particle sampler of the nonlinear diffusion (4) without some kind of regularization.
- The estimation of sparse high dimensional covariance matrices using a small number of independent samples cannot readily be achieved without incorporating some information on the sparsity structure of the desired limit. Several regularization techniques have been developed in the statistics literature; see e.g. [61, 64, 59, 57, 55, 60, 63, 56, 65, 62, 58, 54]. One key common feature is to eliminate (typically long-range) noisy-type empirical correlations when its known that the limiting correlation is null or very small. A common feature of this type of regularization is a projection of the sample covariance into some space of matrices that captures the true sparsity structure.

Consider the first class of perturbation model. Under this model, several variance inflation methods have been proposed in the data assimilation literature as an initial, and simple idea, to address some of these numerical issues [14, 20, 16, 25, 17]. One common feature is to increase the regularity of the sample covariance by increasing the importance of the observation-driven diffusion term. This can be done in several ways: By far the simplest technique is to add an artificial diagonal (positive-definite) matrix to the sample covariance matrix p_t in (6). Another strategy is to artificially increase the spread of the particle system by introducing some nonlinear repulsion term around the sample averages. These two strategies are discussed in Section 4.1 and Section 4.2.

In view of (7), (8), a simple variance inflation method may yield the following Riccati evolution

$$\begin{aligned}
\partial_t P_t^\pi &= \text{Ricc}^\pi(P_t^\pi) \\
&:= [A - \pi(P_t^\pi)S] P_t^\pi + P_t^\pi [A - \pi(P_t^\pi)S]' + R + \pi(P_t^\pi)S\pi(P_t^\pi) \\
&= \text{Ricc}(P_t^\pi) + \Gamma_\pi(P_t^\pi)
\end{aligned} \tag{10}$$

with the quadratic positive mapping Γ_π defined by

$$\Gamma_\pi : Q \in \mathbb{S}_r^+ \mapsto \Gamma_\pi(Q) = \Delta(Q)S\Delta(Q) \quad \text{when} \quad \pi(Q) := Q + \Delta(Q)$$

Obviously, such artificial inflations introduce an extra bias in the particle estimates delivered by the **EnKF** (beyond the bias caused by a finite sample size and (nonlinear) interacting particles). A non-vanishing inflation term will generally be the sole cause of bias in the limiting **EnKF** empirical mean and covariance as $N \rightarrow \infty$.

Later, we consider more general perturbation mappings Γ_π that may arise in scenarios outside (ensemble) Kalman filtering such as in differential games, or in the control of linear stochastic jump systems, etc. These applications were briefly referenced in the introduction. These models will capture the preceding perturbation map as a special case.

Analysis of any bias-variance relationship off requires one to quantify somewhat these two terms. This work focuses largely on the bias, in particular as it follows from the mapping π . For example, under the **EnKF**, the \mathbb{L}_2 -error estimate at the origin with respect to the Frobenius norm is given by

$$\mathbb{E} [\|\pi(p_0) - P_0\|_F^2] = \|\pi(P_0) - P_0\|_F^2 + \mathbb{E} [\|\pi(p_0) - \pi(P_0)\|_F^2]$$

We check this formula via the unbiased property $\mathbb{E}(p_0) = P_0$ of the initial sample covariance. Unfortunately, this unbiasedness property is not preserved in time $t > 0$, due to the mean field interaction of the samples. Indeed, the estimate p_t of P_t arising from the **EnKF** is biased in any

case (e.g. even with $\pi = id$) due to the particle interactions. We don't study the bias arising from the mean field approximation here since our analysis is mostly deterministic and focused on the relevant regularisation mappings. Both the variance and bias resulting from EnKF-type mean field approximations will be the subject of a companion paper.

The general class of perturbation-type mappings considered in this work is discussed in Section 2.2 and Section 3.1 (see also Sections 4.1, 4.2 and 4.5).

Consider now the second class of perturbation models. Under the EnKF framework, these latter projections are often defined in terms of the Hadamard product (a.k.a. Schur product) of the sample covariance matrix with a matrix with $\{0, 1\}$ -valued entries. The null entries represent the desired sparsity topology of the estimate. In the signal processing and data assimilation literature, these projections are often referred to as localization techniques. To avoid the introduction of a huge bias, some prior knowledge of the sparsity structure of the solution of the Riccati equation is needed. However, the sparsity structure of a prescribed filtering problem is generally difficult to extract from the signal and sensor models etc. In some cases, the sparsity structure of the matrices P_t can be estimated online from the particle model; e.g. see the **Isomap** algorithm described in [41, 43]. Section 4.3 presents a block-diagonal filtering problem for which the sparsity structure of the Riccati equation is known.

As with the first class of perturbation models, the choice of mapping π under the second class of projection model also impacts both the bias and the variance of the estimate. The bias term in general will depend on the structure of the initial covariance matrix P_0 , the designer's knowledge of this structure, and the chosen mapping. In the filtering problem discussed in Section 4.3, P_0 is a block-diagonal covariance matrix associated with n -independent filtering problems. In this case, we have $\pi(P_0) = L \odot P_0 = P_0$ for some judicious block-diagonal matrix L with $\{0, 1\}$ -valued entries. With this choice, it also follows that $L \odot P_t = P_t$. However, note the EnKF derived (finite) sample covariance matrices are always biased (due to the particle interactions), so that $L \odot p_t \neq p_t$ for any $t > 0$; hence its effect in practice is to 'enforce' some structure on the sample covariance at every time. In the limit $N \rightarrow \infty$ one would recover the property $L \odot p_t \rightarrow p_t$.

In the statistical literature, a random matrix $L \odot p_0$ associated with some sample covariance matrix p_0 is called a masked or banded sample covariance estimator of some limiting covariance P_0 and the matrix L is interpreted as a mask [55, 60, 65, 58]. In the data assimilation literature, the matrix L is sometimes called the taper matrix. These projection techniques require the solution of the Riccati equation (the desired limit of the sample covariance) to lie within some class of (at least) "approximately band-able" covariance matrices.

The fluctuations of $L \odot p_0$ around its limiting average value $L \odot P_0$ depend only on the non-zero entries. More precisely, for any symmetric mask-matrix L with $\{0, 1\}$ -entries with at most l -zeros in each row we have the Levina-Vershynin's inequality

$$\mathbb{E} [\|L \odot (p_0 - P_0)\|_2] \leq c \log^3(2r) \left[\frac{l}{N} + \sqrt{\frac{l}{N}} \right] \|P_0\|_2$$

for some finite universal constant $c < \infty$; see [65, 58].

The general class of projection-type mappings considered in this work is discussed in Section 2.3 and Section 3.2, including projections onto the Bose-Mesner algebra (see also Section 4.3 and 4.4).

1.3 Statement of the main results

To describe with some precision the main results presented in this article we need to introduce some terminology; see also the notation introduced in Section 1.4.

Definition 1.1. We let $\theta_{s,t}(x)$ be the stochastic flow associated with the underlying signal process (1). We let $\phi_{s,t}(Q)$ be the semigroup associated with the Riccati equation (2),(3). And we let $\psi_{s,t}(x, Q)$ and $\bar{\psi}_{s,t}(x, Q)$ be the stochastic flows associated with the Kalman-Bucy filter and the nonlinear diffusion defined in (2) and (4), with $s \leq t$ and $(x, Q) \in \mathbb{R}^r \times \mathbb{S}_r^+$.

Given some mapping π from \mathbb{S}_r^+ into itself, we let $\phi_{s,t}^\pi(Q)$, resp. $\psi_{s,t}^\pi(x, Q)$ and $\bar{\psi}_{s,t}^\pi(x, Q)$ be the semigroup, respectively the stochastic flows associated with the Riccati equation (8), respectively the Kalman-Bucy filter and the Kalman-Bucy diffusion associated with the nonlinear model (7), with $s \leq t$ and $(x, Q) \in \mathbb{R}^r \times \mathbb{S}_r^+$.

In Section 2.2 and Section 2.3, (cf. Theorem 2.4 and formula (41)), we will check that

$$\phi_t^\pi(Q) \geq \phi_t(Q)$$

This property shows that any π -perturbation of the Kalman-Bucy diffusion induces a larger covariance matrix w.r.t. the Loewner order.

The extra-quadratic operator Γ_π in (10) already hints that the analysis of the semigroups ϕ_t^π is a delicate mathematical problem, since it cannot be deduced directly from that of the Riccati flow ϕ_t . By the Cauchy-Lipschitz theorem, the existence and the uniqueness of the flow of matrices $\phi_t^\pi(Q)$ for any starting covariance matrix Q is ensured by the local Lipschitz property of the drift function Ric^π , on some open interval that may depend on Q . The existence of global solutions on the real line is not ensured as the quadratic term may induce a blow up on some finite time horizon.

Our first contribution concerns the continuity properties of the first class of perturbation models presented in (9) and introduced more formally in Section 2.2. We consider a compact subset Π of continuous mappings $\pi : \mathbb{S}_r^+ \mapsto \mathbb{S}_r^+$ equipped with the uniform norm induced by the \mathbb{L}_2 -norm on \mathbb{S}_r^+ . We let $B(\delta)$ be a δ -ball around the identity mapping. In this notation, our first main result takes the following (mildly informal) form.

Theorem 1.2. Assume that the filtering problem is observable and controllable. In this situation, under some regularity conditions, there exists some $\delta > 0$ such that for any $\epsilon < \delta$, any $\pi \in B(\epsilon)$, and any $n \geq 1$ we have the uniform estimates

$$\sup_{t \geq 0} \|\phi_t^\pi(Q) - \phi_t(Q)\|_2 \leq c(\delta) \epsilon \quad \text{and} \quad \sup_{t \geq 0} \mathbb{E} [\|\psi_{0,t}^\pi(x, Q) - \psi_{0,t}(x, Q)\|_2^{2n}]^{\frac{1}{2n}} \leq c(\delta) \sqrt{n} \epsilon \quad (11)$$

for some finite constant $c(\delta)$ whose values only depend on the parameter δ .

The proof of the Riccati estimates in the l.h.s. of (11) is provided in Section 2.2.2, dedicated to the boundedness and the robustness properties of Riccati semigroups (cf. Theorem 2.6). The proof of the r.h.s. estimates in (11) is provided in Section 3.1 dedicated to the continuity properties of Kalman-Bucy stochastic flows (cf. Theorem 3.2).

Our second objective, given the first class of perturbations, is to quantify the difference between the conditional distributions

$$\eta_{s,t}(x, Q) := \text{Law}(\bar{\psi}_{s,t}(x, Q) \mid \mathcal{F}_{s,t}) \quad \text{and} \quad \eta_{s,t}^\pi(x, Q) := \text{Law}(\bar{\psi}_{s,t}^\pi(x, Q) \mid \mathcal{F}_{s,t})$$

where $\mathcal{F}_{s,t} = \sigma(Y_u, s \leq u \leq t)$ stands for the σ -field generated by the observations from time s to the time horizon t . Our main result takes informally the following form.

Theorem 1.3. Under the assumptions of Theorem 1.2, for any $n \geq 1$, we have the almost sure relative entropy and Wasserstein estimates

$$\begin{aligned} \text{Ent}(\eta_{s,t}^\pi(x, Q) \mid \eta_{s,t}(x, Q)) &\leq c \left[\|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \sqrt{r} \|\phi_{s,t}(Q) - \phi_{s,t}^\pi(Q)\|_2 \right] \\ \mathbb{W}_{2n}[\eta_{s,t}^\pi(x, Q), \eta_{s,t}(x, Q)] &\leq \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2 + c \sqrt{nr} \|\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)\|_2 \end{aligned}$$

for some constant $c < \infty$.

The proof of these estimates, with a more precise description of the constant c is provided in Section 3.1 (see Theorem 3.3 and Theorem 3.5).

The impact of these two theorems is illustrated in Section 4.1, Section 4.2 and Section 4.5 in terms of the variance inflation, mean-repulsion, and the Stein-Shrinkage methods commonly seen in the data assimilation literature.

Our second contribution concerns the continuity properties of the second class of projection mappings presented in (9) and discussed further in Section 2.3. We assume that π is some positive map from \mathbb{M}_r into itself, of the form

$$\pi(Q) = \operatorname{argmin}_{B \in \mathcal{B}} \pi[(Q - B)(Q - B)'] \quad \text{for some matrix ring } \mathcal{B} \subset \mathbb{M}_r$$

From the geometrical viewpoint these orthogonal projections maps the set of \mathbb{S}_r^+ into the set of matrices with the same sparsity structure as the matrices in the ring \mathcal{B} . These projection techniques are unbiased as soon as the *covariance graph* of the filtering model reflecting the sparsity structure of the matrices P_t is defined in terms of the same association scheme. Thus, the (optimal) use of these projections requires some prior knowledge on the sparsity structure of the solution to the Riccati equation.

These models encapsulate most of the localization techniques developed in the data assimilation literature based on Hadamard-Schur product-type projections. For example, the prototype of models satisfying these conditions are orthogonal projections onto the set of block-diagonal matrices $\mathcal{B} = \mathcal{M}_{r[1]} \oplus \dots \oplus \mathcal{M}_{r[n]} \subset \mathbb{M}_r$, with $r = \sum_{1 \leq q \leq n} r[q]$. Another important class of models satisfying the above conditions are orthogonal projections on Bose-Mesner-type cellular algebras w.r.t. the Frobenius norm [46]. These more sophisticated projections are interesting and can be used to project sample covariance matrices based on the topological/graph structure of the matrices (A, R, S) .

See Section 4.3 for applications to block-diagonal masking matrices and Section 4.4 for further discussion on Bose-Mesner projections; e.g. Section 4.4.4 provides an explicit solution of the Riccati equation as soon as the matrices (A, R, S) and the initial condition belong to some Bose-Mesner algebra.

In this context, our third main result takes informally the following form.

Theorem 1.4. *Assume that $(A, A', S, R) \in \mathcal{B}$. In this situation we have*

$$\phi_t^\pi \circ \pi = \phi_t \circ \pi \quad \text{and} \quad \psi_{s,t}^\pi(x, Q) = \psi_{s,t}(x, Q) + [\psi_{s,t}(x, \pi(Q)) - \psi_{s,t}(x, Q)] \quad (12)$$

for any $(x, Q) \in (\mathbb{R}^r \times \mathbb{S}_r^+)$ and $t \geq 0$. In addition, there exists some $\rho > 0$ such that for any $Q \in \mathbb{S}_r^+$ and any time horizon $t \geq 0$ we have the local exponential-Lipschitz inequality

$$\|\phi_t^\pi(Q) - \phi_t(Q)\|_2 \leq c_Q e^{-\rho t} \|Q - \pi(Q)\|_2 \quad (13)$$

for some finite constant c_Q whose values only depend on $\|Q\|_2$.

The l.h.s. of (12) shows that the set \mathcal{B} is stable w.r.t. the π -projected Riccati flow. The r.h.s. of (12) and the exponential estimate (13) shows that, for any initial condition, the Kalman-Bucy stochastic flow as well as the π -projected Riccati flow converges to the set \mathcal{B} as the time horizon t tends to ∞ .

The proofs of the l.h.s. semigroup formula in (12) and the exponential estimate are provided in Section 2.3.1, dedicated to exponential concentration inequalities of the semigroups ϕ_t^π (see Corollary 2.14). The proof of the r.h.s. semigroup formula in (12) is provided in Section 3.2.

Last, but not least, Theorem 1.4 allows one to transfer, without further work, all the exponential contraction inequalities developed in [2], dedicated to the stability properties of Kalman-Bucy diffusions.

1.4 Some basic notation

This section details some basic notation and terms used throughout the article.

Let $\|\cdot\|_2$ be the Euclidean norm on \mathbb{R}^r , $r \geq 1$. We denote by \mathbb{M}_r the set of $(r \times r)$ -square matrices with real entries, $\mathbb{S}_r \subset \mathbb{M}_r$ the set of $(r \times r)$ real symmetric matrices, and by $\mathbb{S}_r^+ \subset \mathbb{S}_r$ the subset of symmetric positive (semi)-definite matrices. With a slight abuse of notation, we denote by Id the $(r \times r)$ standard identity matrix (with the size obvious from the context). Given some subsets $\mathcal{I}, \mathcal{J} \subset \{1, \dots, r\}$ we set $A_{\mathcal{I}, \mathcal{J}} = (A_{i,j})_{(i,j) \in (\mathcal{I} \times \mathcal{J})}$ and $A_{\mathcal{I}} = A_{\mathcal{I}, \mathcal{I}}$.

Denote by $\lambda_i(A)$, with $1 \leq i \leq r$, the non-increasing sequence of eigenvalues of a $(r \times r)$ -matrix A and let $\text{Spec}(A)$ be the set of all eigenvalues. We often denote by $\lambda_{\min}(A) = \lambda_r(A)$ and $\lambda_{\max}(A) = \lambda_1(A)$ the minimal and the maximal eigenvalue. We set $A_{\text{sym}} := (A + A')/2$ for any $(r \times r)$ -square matrix A . We define the logarithmic norm $\mu(A)$ of an $(r_1 \times r_1)$ -square matrix A by

$$\begin{aligned} \mu(A) &:= \inf \{ \alpha : \forall x, \langle x, Ax \rangle \leq \alpha \|x\|_2^2 \} \\ &= \lambda_{\max}(A_{\text{sym}}) \\ &= \inf \{ \alpha : \forall t \geq 0, \|\exp(At)\|_2 \leq \exp(\alpha t) \} \end{aligned} \tag{14}$$

The above equivalent formulations show that

$$\mu(A) \geq \varsigma(A) := \max \{ \text{Re}(\lambda) : \lambda \in \text{Spec}(A) \}$$

where $\text{Re}(\lambda)$ stands for the real part of the eigenvalues λ . The parameter $\varsigma(A)$ is often called the spectral abscissa of A . Also notice that A_{sym} is negative semi-definite as soon as $\mu(A) < 0$. The Frobenius matrix norm of a given $(r_1 \times r_2)$ matrix A is defined by

$$\|A\|_F^2 = \text{tr}(A'A) \quad \text{with the trace operator } \text{tr}(\cdot).$$

If A is a matrix $(r \times r)$, we have $\|A\|_F^2 = \sum_{1 \leq i, j \leq r} A(i, j)^2$. For any $(r \times r)$ -matrix A , we recall norm equivalence formulae

$$\|A\|_2^2 = \lambda_{\max}(A'A) \leq \text{tr}(A'A) = \|A\|_F^2 \leq r \|A\|_2^2$$

For any matrices A and B we also have the estimate

$$\lambda_{\min}(AA')^{1/2} \|B\|_F \leq \|AB\|_F \leq \lambda_{\max}(AA')^{1/2} \|B\|_F$$

We also quote a Lipschitz property of the square root function on (symmetric) definite positive matrices. For any $Q_1, Q_2 \in \mathbb{S}_r^+$

$$\|Q_1^{1/2} - Q_2^{1/2}\| \leq \left[\lambda_{\min}^{1/2}(Q_1) + \lambda_{\min}^{1/2}(Q_2) \right]^{-1} \|Q_1 - Q_2\| \tag{15}$$

for any unitary invariant matrix norm (such as the \mathbb{L}_2 -norm or the Frobenius norm). See for instance Theorem 6.2 on page 135 in [38], as well as Proposition 3.2 on page 591 in [42].

The Hadamard-Schur product of two $(r \times r')$ -matrices A and B of the same size is defined by the matrix $A \odot B$ with entries $(A \odot B)_{i_1, i_2} = A_{i_1, i_2} B_{i_1, i_2}$ for any $1 \leq i_1 \leq r$ and $1 \leq i_2 \leq r'$. With a slight abuse of notation, we denote by J the $(r \times r')$ Hadamard-Schur identity matrix with all unit entries. By Theorem 17 in [39], we recall that for any symmetric positive semi-definite matrices (A, B, P, Q) we have

$$P \geq Q \geq 0 \quad \text{and} \quad A \geq B \geq 0 \quad \implies \quad P \odot A \geq Q \odot B \tag{16}$$

Now, given some random variable Z with some probability measure or distribution η and some measurable function f on some product space \mathbb{R}^r , we let

$$\eta(f) = \mathbb{E}(f(Z)) = \int f(x) \eta(dx)$$

be the integral of f w.r.t. η or the expectation of $f(Z)$. As a rule any multivariate variable, say Z , is represented by a column vector and we use the transposition operator Z' to denote the row vector (similarly for matrices; already seen above).

We also need to consider the n -th Wasserstein distance between two probability measures ν_1 and ν_2 on \mathbb{R}^r defined by

$$\mathbb{W}_n(\nu_1, \nu_2) = \inf \left\{ \mathbb{E}(\|Z_1 - Z_2\|_2^n)^{\frac{1}{n}} \right\}$$

The infimum in the above formula is taken over all pairs of random variable (Z_1, Z_2) such that $\text{Law}(Z_i) = \nu_i$, with $i = 1, 2$. We denote by $\text{Ent}(\nu_1 \mid \nu_2)$ the Boltzmann-relative entropy

$$\text{Ent}(\nu_1 \mid \nu_2) := \int \log \left(\frac{d\nu_1}{d\nu_2} \right) d\nu_1 \quad \text{if } \nu_1 \ll \nu_2, \text{ and } +\infty \text{ otherwise.}$$

1.5 Some background and preliminary results

1.5.1 Observability, controllability and the steady-state Riccati equation

We assume that $(A, R^{1/2})$ is a controllable pair and (A, C) is observable in the sense that

$$\left[R^{1/2}, A(R^{1/2}), \dots, A^{r-1}R^{1/2} \right] \quad \text{and} \quad \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \end{bmatrix} \quad (17)$$

have rank r . We consider the observability and controllability Gramians $(\mathcal{O}_t, \mathcal{C}_t(\mathcal{O}))$ and $(\mathcal{C}_t, \mathcal{O}_t(\mathcal{C}))$ associated with the triplet (A, R, S) and defined by

$$\begin{aligned} \mathcal{O}_t &:= \int_0^t e^{-A's} S e^{-As} ds & \text{and} & \quad \mathcal{C}_t(\mathcal{O}) := \mathcal{O}_t^{-1} \left[\int_0^t e^{-(t-s)A'} \mathcal{O}_s R \mathcal{O}_s e^{-(t-s)A} ds \right] \mathcal{O}_t^{-1} \\ \mathcal{C}_t &:= \int_0^t e^{As} R e^{A's} ds & \text{and} & \quad \mathcal{O}_t(\mathcal{C}) := \mathcal{C}_t^{-1} \left[\int_0^t e^{(t-s)A} \mathcal{C}_s S \mathcal{C}_s e^{(t-s)A} ds \right] \mathcal{C}_t^{-1} \end{aligned}$$

Given the rank assumptions on (17), there exists some parameters $v, \varpi_{\pm}^{o,c}, \varpi_{\pm}^c(\mathcal{O}), \varpi_{\pm}^o(\mathcal{C}) > 0$ such that

$$\varpi_-^c Id \leq \mathcal{C}_v \leq \varpi_+^c Id \quad \text{and} \quad \varpi_-^o Id \leq \mathcal{O}_v \leq \varpi_+^o Id \quad (18)$$

as well as

$$\varpi_-^c(\mathcal{O}) Id \leq \mathcal{C}_v(\mathcal{O}) \leq \varpi_+^c(\mathcal{O}) Id \quad \text{and} \quad \varpi_-^o(\mathcal{C}) Id \leq \mathcal{O}_v(\mathcal{C}) \leq \varpi_+^o(\mathcal{C}) Id$$

The parameter v is often called the interval of observability-controllability. By Theorem 4.4 in [2], for any $t \geq v$ and any $Q \in \mathbb{S}_r^+$ we have the uniform estimates

$$(\mathcal{O}_v(\mathcal{C}) + \mathcal{C}_v^{-1})^{-1} \leq \phi_t(Q) \leq \mathcal{O}_v^{-1} + \mathcal{C}_v(\mathcal{O}) \quad (19)$$

When (18) is satisfied, we say that a triplet (A, R, S) satisfy the Gramian condition for some parameters $v, \varpi_{\pm}^{o,c} > 0$. These conditions ensure the existence and the uniqueness of a positive-definite fixed-point matrix P solving the so-called algebraic Riccati equation

$$\text{Ricc}(P) := AP + PA' - PSP + R = 0. \quad (20)$$

Importantly, in this case, the matrix difference $A - PS$ is asymptotically stable even when the signal matrix A is unstable. See the discussion and linked references in [2] for further discussion on the nominal Riccati equation.

1.5.2 Exponential and Kalman-Bucy semigroup estimates

The transition matrix associated with a smooth flow of $(r \times r)$ -matrices $A : u \mapsto A_u$ is denoted by

$$\mathcal{E}_{s,t}(A) = \exp \left[\oint_s^t A_u \, du \right] \iff \partial_t \mathcal{E}_{s,t}(A) = A_t \mathcal{E}_{s,t}(A) \quad \text{and} \quad \partial_s \mathcal{E}_{s,t}(A) = -\mathcal{E}_{s,t}(A) A_s$$

for any $s \leq t$, with $\mathcal{E}_{s,s} = Id$, the identity matrix. Equivalently in terms of the fundamental solution matrices $\mathcal{E}_t(A) := \mathcal{E}_{0,t}(A)$ we have $\mathcal{E}_{s,t}(A) = \mathcal{E}_t(A) \mathcal{E}_s(A)^{-1}$.

The following technical lemma provides a pair of semigroup estimates of the state transition matrices associated with a sum of drift-type matrices.

Lemma 1.5 ([2]). *Let $A : u \mapsto A_u$ and $B : u \mapsto B_u$ be some smooth flows of $(r \times r)$ -matrices. For any $s \leq t$ and any matrix norm $\|\cdot\|$ we have*

$$\|\mathcal{E}_{s,t}(A)\| \leq \alpha_A \exp(-\omega_A(t-s)) \Rightarrow \|\mathcal{E}_{s,t}(A+B)\| \leq \alpha_A \exp \left[-\omega_A(t-s) + \alpha_A \int_s^t \|B_u\| \, du \right]$$

For any $s \leq t$ and $Q \in \mathbb{S}_r^+$ we set

$$E_{s,t}(Q) := \exp \left[\oint_s^t (A - \phi_u(Q)S) \, du \right]$$

When $s = 0$ sometimes we write $E_t(Q)$ instead of $E_{0,t}(Q)$. In this notation we have

$$E_{s,t}(Q) = E_t(Q) E_s(Q)^{-1}$$

For any $s \leq u \leq t$ and $Q \in \mathbb{S}_r^+$ we set

$$E_{t|s}(Q) = \exp \left[\oint_s^t (A - \phi_{s,v}(Q)S) \, dv \right] \quad \text{and} \quad E_{u,t|s}(Q) := E_{t|s}(Q) E_{u|s}(Q)^{-1}$$

Also observe that

$$E_{s,t}(Q) = \exp \left[\oint_s^t (A - \phi_{s,u}(\phi_s(Q))S) \, du \right] = E_{t|s}(\phi_s(Q))$$

For any $s \leq u \leq t$ and any $Q \in \mathbb{S}_r^+$ we have

$$E_{t|s}(Q) = E_{t-s}(Q) \quad \text{and} \quad E_{u,t|s}(Q) = E_{(u-s),(t-s)}(Q) \quad (21)$$

Observe that the Riccati equation is time-homogeneous so that

$$\phi_{s,s+t}(Q) = \phi_t(Q) := \phi_{0,t}(Q).$$

By Proposition 4.3 in [2] we have

$$0 \leq \phi_t(Q) \leq P + e^{(A-PS)t}(Q - P)e^{(A-PS)t} \implies \|\phi_t(Q)\|_2 \leq \|P\|_2 + \kappa\|Q - P\|_2 \quad (22)$$

for some constant κ whose values doesn't depend on the time parameter nor on Q .

Theorem 1.6 ([2]). *For any $Q_1, Q_2 \in \mathbb{S}_r^+$ and for any $t \geq 0$ we have the local contraction inequality*

$$\|E_t(Q_1)\|_2 \leq \kappa_E(\|Q_1\|_2) e^{-2\nu t} \quad (23)$$

$$\|\phi_t(Q_2) - \phi_t(Q_1)\|_2 \leq \kappa_\phi(\|Q_1\|_2, \|Q_2\|_2) e^{-2\nu t} \|Q_2 - Q_1\|_2 \quad (24)$$

$$\|E_t(Q_2) - E_t(Q_1)\|_2 \leq \kappa_E(\|Q_1\|_2, \|Q_2\|_2) e^{-\nu t} \|Q_2 - Q_1\|_2 \quad (25)$$

for some rate $\nu > 0$, and some finite non-decreasing functions $\kappa_E(q_1), \kappa_E(q_1, q_2), \kappa_\phi(q_1, q_2) < \infty$.

A proof of this theorem can be found in [2, see e.g. Corollary 4.10 and 4.13].

2 Riccati semigroups

2.1 Variational and backward semigroups

We let $\mathcal{L}(\mathbb{S}_r, \mathbb{S}_r)$ be the set of bounded linear functional from \mathbb{S}_r into itself, and equipped with the Frobenius norm. A mapping $\phi : \mathbb{S}_r^+ \mapsto \mathbb{S}_r^+$ is Fréchet differentiable at some $Q_1 \in \mathbb{S}_r^+$ if there exists a continuous linear functional $\partial\phi(Q_1) \in \mathcal{L}(\mathbb{S}_r, \mathbb{S}_r)$ such that

$$\lim_{Q_2 \rightarrow Q_1} \|Q_2 - Q_1\|_F^{-1} \|\phi(Q_2) - \phi(Q_1) - \partial\phi(Q_1) \cdot (Q_2 - Q_1)\|_F = 0$$

For instance the first-order Frechet-derivative of the Riccati quadratic drift function

$$\text{Ricc} : Q \in \mathbb{S}_r^+ \mapsto \text{Ricc}(Q) \in \mathbb{S}_r$$

defined in (3) is given for any $(Q_1, Q_2) \in (\mathbb{S}_r^+ \times \mathbb{S}_r)$ by the formula

$$\partial\text{Ricc}(Q_1) \cdot Q_2 = (A - Q_1 S)Q_2 + Q_2(A - Q_1 S)' \quad (26)$$

Lemma 2.1. *For any $t \geq 0$ the mapping $Q \mapsto \phi_t(Q)$ is Fréchet differentiable and for any $(Q_1, Q_2) \in (\mathbb{S}_r^+ \times \mathbb{S}_r^+)$ we have the formulae*

$$\partial\phi_t(Q_1) \cdot Q_2 = E_t(Q_1) Q_2 E_t(Q_1)'$$

Proof. Using the decomposition

$$\begin{aligned} \phi_t(Q_1) - \phi_t(Q_2) &= E_{s,t}(Q_2) [\phi_s(Q_1) - \phi_s(Q_2)] E_{s,t}(Q_2)' \\ &\quad - \int_s^t E_{u,t}(Q_2) [\phi_u(Q_1) - \phi_u(Q_2)] S [\phi_u(Q_1) - \phi_u(Q_2)] E_{u,t}(Q_2)' du \end{aligned}$$

we have

$$\begin{aligned} \phi_t(Q_2) - \phi_t(Q_1) &= E_t(Q_1) [Q_2 - Q_1] E_t(Q_1)' \\ &\quad - \int_0^t E_{u,t}(Q_1) [\phi_u(Q_2) - \phi_u(Q_1)] S [\phi_u(Q_2) - \phi_u(Q_1)] E_{u,t}(Q_1)' du \end{aligned}$$

We end the proof of the first assertion using the Lipschitz property (24). The proof of the lemma is completed. \blacksquare

We have the following backward flow and first-order variational result that will be used subsequently, but which is also of interest in its own right.

Proposition 2.2. *For any $Q \in \mathbb{S}_r^+$ and any $0 \leq s \leq t$ we have*

$$\partial_s \phi_{s,t}(Q) = -\text{Ricc}(\phi_{s,t}(Q)) \quad \text{and} \quad \partial_t \phi_{s,t}(Q) = \text{Ricc}(\phi_{s,t}(Q)) = \partial \phi_{s,t}(Q) \cdot \text{Ricc}(Q)$$

In addition, the first-order variational equation associated with the Riccati equation is given by the composition formula

$$\partial_t (\partial \phi_t(Q)) = \partial \text{Ricc}(\phi_t(Q)) \circ \partial \phi_t(Q) \quad (27)$$

Proof. For any $Q \in \mathbb{S}_r^+$ we have

$$\partial_s \phi_{s,t}(Q) = \partial_s \phi_{0,t-s}(Q) = -\text{Ricc}(\phi_{0,t-s}(Q)) = -\text{Ricc}(\phi_{s,t}(Q))$$

On the other hand, we have

$$\begin{aligned} & \|\text{Ricc}(\phi_{s-h,u}(Q)) - \text{Ricc}(Q)\|_F \leq c_Q h \\ \implies & \left\| \int_{s-h}^s [\text{Ricc}(\phi_{s-h,u}(Q)) - \text{Ricc}(Q)] du \right\|_F \leq c_Q h^2 \end{aligned}$$

for some finite constant c_Q whose values only depends on $\|Q\|_F$. Using Lemma 2.1 this yields

$$\begin{aligned} \phi_{s-h,t}(Q) - \phi_{s,t}(Q) &= \phi_{s,t}(\phi_{s-h,s}(Q)) - \phi_{s,t}(Q) \\ &= \phi_{s,t} \left(Q + \int_{s-h}^s \text{Ricc}(\phi_{s-h,u}(Q)) du \right) - \phi_{s,t}(Q) \\ &= \partial \phi_{s,t}(Q) \cdot \left[\int_{s-h}^s \text{Ricc}(\phi_{s-h,u}(Q)) du \right] + o(h) \\ &= \partial \phi_{s,t}(Q) \cdot \text{Ricc}(Q) h \\ &\quad + \partial \phi_{s,t}(Q) \cdot \left[\int_{s-h}^s [\text{Ricc}(\phi_{s-h,u}(Q)) - \text{Ricc}(Q)] du \right] + o(h) \\ &= \partial \phi_{s,t}(Q) \cdot \text{Ricc}(Q) h + o(h) \end{aligned}$$

This implies that

$$\partial_s \phi_{s,t}(Q) = \lim_{h \rightarrow 0} \frac{1}{-h} [\phi_{s-h,t}(Q) - \phi_{s,t}(Q)] = \partial \phi_{s,t}(Q) \cdot \text{Ricc}(Q)$$

from which we conclude that

$$-\text{Ricc}(\phi_{s,t}(Q)) + \partial \phi_{s,t}(Q) \cdot \text{Ricc}(Q) = \partial_s \phi_{s,t}(Q) + \partial_t \phi_{s,t}(Q) = 0 \quad (28)$$

Finally, by Lemma 2.1 and (26) we have

$$\begin{aligned} \partial_t [\partial \phi_t(Q_1) \cdot Q_2] &= [A - \phi_t(Q_1)S] [\partial \phi_t(Q_1) \cdot Q_2] + [\partial \phi_t(Q_1) \cdot Q_2] [A - \phi_t(Q_1)S]' \\ &= \partial \text{Ricc}(\phi_t(Q_1)) \cdot [\partial \phi_t(Q_1) \cdot Q_2] \\ &= [\partial \text{Ricc}(\phi_t(Q_1)) \circ \partial \phi_t(Q_1)] (Q_2) \end{aligned}$$

This ends the proof of the proposition. \blacksquare

2.2 Perturbation-type models

2.2.1 First and second order perturbations

We consider perturbation-type distortions in (8) of the first type in (9). In particular, we consider a class of perturbation mappings Γ_π , see (10), but in a more general form than previously discussed in that section (which focused on **EnKF** drivers). It is here that we also capture those perturbations relevant in, e.g., linear-quadratic differential games, control of stochastic jump processes, robust control theory, etc. Consider (8),(10) and going forward we assume that Γ_π has the form

$$(H)_0 \quad \Gamma_\pi(Q) = B_0 + B_1 Q + Q B_1' + Q B_2 Q + \mathcal{R}(Q)$$

for some given matrices $(B_0, B_1, B_2) \in \mathbb{S}_r^3$ such that $B_2 \leq S$, and a uniformly bounded (symmetric) remainder term

$$\varpi := \sup_{Q \in \mathbb{S}_r^+} \|\mathcal{R}(Q)\|_2 < \infty \implies \mathcal{R}(Q) \leq \varpi Id$$

In this situation, the π -Riccati drift function Ricc^π takes the form

$$\text{Ricc}^\pi(Q) = \text{Ricc}_\pi(Q) + \mathcal{R}_\pi(Q) \leq \text{Ricc}_\pi(Q)$$

with

$$\text{Ricc}_\pi(Q) := A_\pi Q + Q A_\pi' + R_\pi - Q S_\pi Q \quad \mathcal{R}_\pi(Q) = \mathcal{R}(Q) - \varpi Id \leq 0 \quad (29)$$

and the matrices

$$R_\pi := R + B_0 + \varpi Id \quad A_\pi := A + B_1 \quad \text{and} \quad S_\pi := S - B_2 \geq 0$$

Definition 2.3. We let $\phi_{\pi,t}$, resp. ϕ_t^π be the Riccati flows associated with the drift function Ricc_π and resp. Ricc^π . We consider the observability and the controllability Gramians $(\mathcal{O}_{\pi,t}, \mathcal{C}_{\pi,t}(\mathcal{O}))$ and $(\mathcal{C}_{\pi,t}, \mathcal{O}_{\pi,t}(\mathcal{C}))$ associated with the triplet (A_π, R_π, S_π) .

We also let Ξ_π be the mapping from \mathbb{S}_r into itself defined by

$$\Xi_\pi(Q) := \text{Ricc}_\pi(Q) - \text{Ricc}(Q) = (A_\pi - A)Q + Q(A_\pi - A)' + (R_\pi - R) - Q(S_\pi - S)Q$$

We also set

$$\gamma(\pi) := \|A_\pi - A\|_2 + \|R_\pi - R\|_2 + \|S_\pi - S\|_2$$

We consider the following condition,

$$(H)_1 \quad (A_\pi, R_\pi, S_\pi) \text{ satisfies the Gramian condition (18) for some } v_\pi, \varpi_\pm^{o,c}(\pi) > 0$$

We recall that this condition ensures the existence and the uniqueness of a positive-definite fixed-point matrix P_π solving the so-called algebraic Riccati equation

$$\text{Ricc}_\pi(P_\pi) := A_\pi P_\pi + P_\pi A_\pi' - P_\pi S_\pi P_\pi + R_\pi = 0. \quad (30)$$

In addition, the matrix difference $A_\pi - P_\pi S_\pi$ is asymptotically stable.

Our first objective is to analyze the existence and the uniqueness of the flow ϕ_t^π .

Theorem 2.4. Assume $(H)_0$ and $(H)_1$. For any $t \geq 0$ and $Q \in \mathbb{S}_r^+$,

$$(\mathcal{O}_v(\mathcal{C}) + \mathcal{C}_v^{-1})^{-1} \leq \phi_t(Q) \leq \phi_t^\pi(Q) \leq \phi_{\pi,t}(Q) \leq \mathcal{O}_{\pi,v_\pi}^{-1} + \mathcal{C}_{\pi,v_\pi}(\mathcal{O}) \quad (31)$$

The upper and lower bound estimates in the r.h.s. and the l.h.s. of (31) are only valid for $t \geq v \vee v_\pi$.

Proof. By (19) and (H)₁ we have the uniform estimates

$$(\mathcal{O}_{\pi, v_\pi}(\mathcal{C}) + \mathcal{C}_{\pi, v_\pi}^{-1})^{-1} \leq \phi_{\pi, t}(Q) \leq \mathcal{O}_{\pi, v_\pi}^{-1} + \mathcal{C}_{\pi, v_\pi}(\mathcal{O})$$

We let $E_{\pi, t|s}(Q)$ be the transition semigroups defined as $E_{t|s}(\phi_s^\pi(Q))$ by replacing (A, ϕ_t) by $(A_\pi, \phi_{\pi, t})$. In this notation, the proof (31) is a direct consequence of the backward perturbation formulae

$$\phi_t^\pi(Q) - \phi_{\pi, t}(Q) = \int_0^t E_{\pi, t|s}(\phi_s^\pi(Q)) \mathcal{R}_\pi[\phi_s^\pi(Q)] E_{\pi, t|s}(\phi_s^\pi(Q))' ds \leq 0 \quad (32)$$

as well as

$$\phi_t^\pi(Q) - \phi_t(Q) = \int_0^t E_{t|s}(\phi_s^\pi(Q)) \Gamma_\pi[\phi_s^\pi(Q)] E_{t|s}(\phi_s^\pi(Q))' ds \geq 0 \quad (33)$$

That is, the l.h.s. estimate in (31) is a direct consequence of (19) and the relationship $\phi_t(Q) \leq \phi_t^\pi(Q) \leq \phi_{\pi, t}(Q)$ following from (32) and (33). The r.h.s. estimate in (31) follows obviously from the above.

To check (33) we use the interpolating path

$$s \in [0, t] \mapsto \phi_{s, t}(\phi_s^\pi(Q)) \quad \text{from} \quad \phi_t(Q) \quad \text{to} \quad \phi_t^\pi(Q)$$

By Proposition 2.2 we have

$$\begin{aligned} \partial_s \phi_{s, t}(\phi_s^\pi(Q)) &= -\text{Ricc}(\phi_{s, t}(\phi_s^\pi(Q))) + \partial \phi_{s, t}(\phi_s^\pi(Q)) \cdot \partial_s \phi_s^\pi(Q) \\ &= \partial \phi_{s, t}(\phi_s^\pi(Q)) \cdot \Gamma_\pi[\phi_s^\pi(Q)] = E_{t|s}(\phi_s^\pi(Q)) \Gamma_\pi[\phi_s^\pi(Q)] E_{t|s}(\phi_s^\pi(Q)) \end{aligned}$$

This ends the proof of (33). The proof of (32) follows the same arguments, thus it is skipped. This ends the proof of the theorem. \blacksquare

The next lemma compares the semigroups $\phi_{\pi, t}(Q)$ and $\phi_t(Q)$ when the matrices (A_π, R_π, S_π) are close to (A, R, S) .

Lemma 2.5. Assume (H)₀ and (H)₁. For any $t \geq 0$ and $Q \in \mathbb{S}_r^+$ we have

$$\phi_{\pi, t}(Q) - \phi_t(Q) = \int_0^t E_{t|s}(\phi_{\pi, s}(Q)) \Xi_\pi[\phi_{\pi, s}(Q)] E_{t|s}(\phi_{\pi, s}(Q))' ds \quad (34)$$

as well as

$$\phi_t(Q) - \phi_{\pi, t}(Q) = \int_0^t E_{\pi, t|s}(\phi_s(Q)) \Xi_\pi[\phi_s(Q)] E_{\pi, t|s}(\phi_s(Q))' ds \quad (35)$$

The proof of this lemma follows the same arguments as the proof of Theorem 2.4; thus it is skipped. Observe that

$$(34) \implies P_\pi - P = \phi_t(P_\pi) - P + \int_0^t E_{t|s}(P_\pi) \Xi_\pi[P_\pi] E_{t|s}(P_\pi)' ds$$

and

$$(35) \implies P - P_\pi = \phi_{\pi, t}(P) - P_\pi + \int_0^t E_{\pi, t|s}(P) \Xi_\pi[P] E_{\pi, t|s}(P)' ds$$

2.2.2 Robustness theorems

We equip the set $\mathcal{C}(\mathbb{S}_r^+, \mathbb{S}_r^+)$ of continuous mappings $\pi : \mathbb{S}_r^+ \mapsto \mathbb{S}_r^+$ with the uniform norm

$$\|\pi_1 - \pi_2\| = \sup_{Q \in \mathbb{S}_r^+} \|\pi_1(Q) - \pi_2(Q)\|_2$$

Let $\Pi \subset \mathcal{C}(\mathbb{S}_r^+, \mathbb{S}_r^+)$ be a compact subset, and let $t > 0$ be some fixed time horizon. For any $\delta > 0$, we let $B(\delta)$ be the δ -ball around the identity mapping; that is

$$B(\delta) := \{\pi \in \Pi : \|\pi - id\| \leq \delta\}$$

We consider the following continuity condition

$$(H)_2 \quad \forall \epsilon, \alpha \in]0, 1] \quad \exists \delta > 0 \quad \text{such that} \quad \forall \pi \in B(\delta) \quad \text{we have}$$

$$\alpha \leq S_\pi \leq \alpha^{-1} S \quad \alpha R \leq R_\pi \leq \alpha^{-1} R \quad \text{and} \quad \|A - A_\pi\|_2 \leq \epsilon$$

The main objective of this section is to prove the following theorem.

Theorem 2.6. *Assume $(H)_0$ and $(H)_2$. In this situation, there exists some $\delta > 0$ such that for any $\pi \in B(\delta)$, any time horizon $t \geq 0$ and any $Q \in \mathbb{S}_r^+$ we have*

$$\|\phi_t^\pi(Q) - \phi_t(Q)\|_2 \leq \|\phi_{\pi,t}(Q) - \phi_t(Q)\|_2 \leq [\chi_1(\delta) + e^{-4t\nu} \chi_2(\delta, \|Q\|_2)] \gamma(\pi)$$

for some finite constant $\chi_1(\delta)$, resp. $\chi_2(\delta, \|Q\|_2)$, whose values only depend on the parameter δ , resp. on $(\delta, \|Q\|)$. In particular we have

$$\forall \pi \in B(\delta) \quad \|P_\pi - P\|_2 \leq \chi_1(\delta) \gamma(\pi)$$

The proof of the theorem relies on the following proposition.

Proposition 2.7. *We have $(H)_2 \Rightarrow (H)_1$. In addition when $(H)_2$ is met, for any $\alpha \in]0, 1]$ there exist some $\delta > 0$ such that the matrices (A_π, R_π, S_π) indexed by mappings π in the δ -ball $B(\delta)$ satisfy the Gramian condition with a common interval of observability-controllability $v_\pi = v$ and some parameters*

$$\alpha \varpi_\pm^{o,c} \leq \varpi_{\pi,\pm}^{o,c} \leq \alpha^{-1} \varpi_\pm^{o,c}$$

and well as

$$\alpha \varpi_\pm^c(\mathcal{O}) \leq \varpi_{\pi,\pm}^c(\mathcal{O}) \leq \alpha^{-1} \varpi_\pm^c(\mathcal{O}) \quad \text{and} \quad \alpha \varpi_\pm^o(\mathcal{C}) \leq \varpi_{\pi,\pm}^o(\mathcal{C}) \leq \alpha^{-1} \varpi_\pm^o(\mathcal{C})$$

We already quote a direct consequence of Theorem 2.4 and Proposition 2.7.

Corollary 2.8. *Assume $(H)_2$. In this situation, for any $\alpha \in]0, 1]$ there exist some $\delta > 0$ such that for any $\pi \in B(\delta)$ and any $t \geq 0$ we have the uniform estimates*

$$(\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c)^{-1} Id \leq \phi_{t+v}(Q) \leq \phi_{t+v}^\pi(Q) \leq \phi_{\pi,t+v}(Q) \leq \alpha^{-1} [\varpi_+^c(\mathcal{O}) + 1/\varpi_-^o] Id \quad (36)$$

The proof of the above proposition relies on a couple of comparison lemmas of interest on their own.

Lemma 2.9. *Let $V_1, V_2 \in \mathbb{S}_r^+$ be a couple of definite positive matrices s.t. $V_1 \geq V_2$. We set*

$$Q_1 := U_1 V_1 U_1' \quad \text{and} \quad Q_2 := U_2 V_2 U_2'$$

for some $(U_1, U_2) \in \mathbb{M}_r^2$. Assume that $Q_2 \geq q_2 \text{Id}$, for some $q_2 > 0$. In this situation, whenever U_2 is invertible we have

$$\|Q_2\|_2 \|U_1 U_2^{-1} - \text{Id}\|_2 < \sqrt{1 + q_2} - 1 \implies Q_1 \geq q_{1,2} Q_2$$

with

$$q_{1,2} = \left[1 - q_2^{-1} \left\{ (1 + \|Q_2\|_2 \|U_1 U_2^{-1} - \text{Id}\|_2)^2 - 1 \right\} \right]$$

Proof. We set

$$U_2 U_1^{-1} = \text{Id} + U_{1,2} \quad \text{and} \quad U_1 U_2^{-1} = \text{Id} + U_{2,1}$$

Observe that

$$\begin{aligned} Q_2 \geq q_2 \text{Id} &\implies \lambda_{\min}(Q_2) \geq q_2 \implies \lambda_{\min}(Q_2^{1/2}) \geq \sqrt{q_2} \\ &\implies \lambda_{\max}(Q_2^{-1/2}) \leq 1/\sqrt{q_2} \implies \|Q_2^{-1/2}\|_2^2 \leq q_2^{-1} \end{aligned}$$

In this situation, we have

$$\begin{aligned} Q_1 &\geq U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' \\ &\implies Q_1^{-1} \leq Q_2^{-1/2} \left[Q_2^{-1/2} U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' Q_2^{-1/2} \right]^{-1} Q_2^{-1/2} \end{aligned}$$

Observe that

$$\begin{aligned} &\left[Q_2^{-1/2} U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' Q_2^{-1/2} \right]^{-1} \\ &= \left[\text{Id} - Q_2^{-1/2} \{ Q_2 - U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' \} Q_2^{-1/2} \right]^{-1} \\ &= \sum_{n \geq 0} \left[Q_2^{-1/2} \{ Q_2 - U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' \} Q_2^{-1/2} \right]^n \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\{ Q_2 - U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' \} \\ &= \{ Q_2 - [\text{Id} + U_{2,1}] Q_2 [\text{Id} + U_{2,1}'] \} = - [U_{2,1} Q_2 + Q_2 U_{2,1}'] - U_{2,1} Q_2 U_{2,1}' \end{aligned}$$

This implies that

$$\begin{aligned} &\|Q_2^{-1/2} \{ Q_2 - U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' \} Q_2^{-1/2}\| \\ &\leq q_2^{-1} \|U_{2,1}\|_2 \|Q_2\|_2 [2 + \|U_{2,1}\|_2 \|Q_2\|_2] = q_2^{-1} \left\{ (1 + \|U_{2,1}\|_2 \|Q_2\|_2)^2 - 1 \right\} < 1 \end{aligned}$$

from which we conclude that

$$\left[Q_2^{-1/2} U_1 U_2^{-1} Q_2 (U_1 U_2^{-1})' Q_2^{-1/2} \right]^{-1} \leq \left[1 - q_2^{-1} \left\{ (1 + \|U_{2,1}\|_2 \|Q_2\|_2)^2 - 1 \right\} \right]^{-1} \text{Id}$$

This yields the estimate

$$Q_1^{-1} \leq q_{1,2}^{-1} Q_2^{-1} \iff Q_1 \geq q_{1,2} Q_2$$

with

$$q_{1,2}^{-1} = \left[1 - q_2^{-1} \left\{ (1 + \|U_{2,1}\|_2 \|Q_2\|_2)^2 - 1 \right\} \right]^{-1}$$

This ends the proof of the lemma. ■

Lemma 2.10. *Let \mathcal{U}, \mathcal{V} be a pair of bounded functions from $[0, t] \times \Pi$ into \mathbb{S}_r^+ . We consider the integral mappings*

$$(s, \pi) \in ([0, t] \times \Pi) \mapsto \mathcal{W}_s(\pi) := \int_0^s \mathcal{U}_u(\pi) \mathcal{V}_u(\pi) \mathcal{U}'_u(\pi) du \in \mathbb{S}_r^+$$

Let $\pi_1, \pi_2 \in \Pi$ be such that

$$\forall s \in [0, t] \quad \mathcal{V}_s(\pi_1) \geq \mathcal{V}_s(\pi_2) \quad \text{and} \quad \mathcal{W}_t(\pi_2) \geq \varpi_{-,t}(\pi_2) Id \quad \text{for some} \quad \varpi_{-,t}(\pi_2) > 0 \quad (37)$$

Also assume that the flow of matrices $\mathcal{U}_s(\pi_2)$ are invertible for any $s \in [0, t]$ and they satisfy the following Lipschitz inequality

$$\sup_{s \in [0, t]} \|\mathcal{U}_s(\pi_1) \mathcal{U}_s(\pi_2)^{-1} - Id\|_2 \leq \text{lip}_t(\mathcal{U}) \|\pi_1 - \pi_2\| \quad (38)$$

for some finite constant $\text{lip}_t(\mathcal{U})$. In this situation, for any $\epsilon \in]0, 1]$ there exists some parameter $\delta = \delta(t, \epsilon, \pi_2) > 0$ such that

$$\|\pi_1 - \pi_2\| \leq \delta \implies \mathcal{W}_t(\pi_1) \geq \epsilon \mathcal{W}_t(\pi_2)$$

The proof of this lemma follows the same line of argument as used in the proof of Lemma 2.9, thus it is skipped here. For the convenience of the reader a detailed proof of the lemma is provided in the appendix on page 42.

Now we come to the proof of Proposition 2.7.

Proof of Proposition 2.7:

We assume that for any $\epsilon \in]0, 1]$ there exists some $\delta > 0$ such that

$$\|\pi - id\| \leq \delta \implies S_\pi \geq (1 - \epsilon) S \quad \text{and} \quad R_\pi \geq (1 - \epsilon) R$$

We apply Lemma 2.10 to the functions

$$\begin{aligned} \mathcal{W}_t^o(\pi) &:= (1 - \epsilon)^{-1} \mathcal{O}_{\pi, t} \implies \mathcal{W}_t^o(id) = (1 - \epsilon)^{-1} \mathcal{O}_t \\ \mathcal{W}_t^c(\pi) &:= (1 - \epsilon)^{-1} \mathcal{C}_{\pi, t} \implies \mathcal{W}_t^c(id) = (1 - \epsilon)^{-1} \mathcal{C}_t \end{aligned}$$

with $(\pi_1, \pi_2) = (\pi, id)$ and the time horizon $t = v$. For any $\epsilon_1 \in]0, 1]$ there exists some parameter $\delta_1 = \delta(\epsilon_1, v)$ such that

$$\|\pi - id\| \leq \delta_1 \implies \mathcal{O}_{\pi, v} \geq \epsilon_1 \mathcal{O}_v \quad \text{and} \quad \mathcal{C}_{\pi, v} \geq \epsilon_1 \mathcal{C}_v$$

We assume that for any $\epsilon \in]0, 1]$ there exists some $\delta > 0$ such that

$$\|\pi - id\| \leq \delta \implies S \geq (1 - \epsilon) S_\pi \quad \text{and} \quad R \geq (1 - \epsilon) R_\pi$$

We apply Lemma 2.10 to the functions

$$\mathcal{W}_t^o(\pi) = \mathcal{O}_{\pi,t} \quad \text{and} \quad \mathcal{W}_t^c(\pi) = \mathcal{C}_{\pi,t}$$

with $(\pi_1, \pi_2) = (id, \pi)$ and the time horizon $t = v$. From previous estimates we have

$$\|\pi_2 - id\| \leq \delta_1 \implies \mathcal{W}_v^o(\pi_2) \geq \epsilon_1 \varpi_-^o Id \quad \text{and} \quad \mathcal{W}_v^c(\pi_2) \geq \epsilon_1 \varpi_-^c Id$$

By Lemma 2.10 for any $\epsilon_2 \in]0, 1]$ we can choose $\delta = \delta(\epsilon_1, \epsilon_2, v)$ such that

$$\|\pi - id\| \leq \delta \implies \mathcal{O}_v \geq \epsilon_2 \mathcal{O}_{\pi,v} \quad \text{and} \quad \mathcal{C}_v \geq \epsilon_2 \mathcal{C}_{\pi,v}$$

This shows that

$$\|\pi - id\| \leq \delta \implies \epsilon_1 \mathcal{O}_v \leq \mathcal{O}_{\pi,v} \leq \epsilon_2^{-1} \mathcal{O}_v \quad \text{and} \quad \epsilon_1 \mathcal{C}_v \leq \mathcal{C}_{\pi,v} \leq \epsilon_2^{-1} \mathcal{C}_v$$

In the same vein we proof the estimates of the Gramians $\mathcal{O}_{\pi,v}(\mathcal{C})$ and $\mathcal{C}_{\pi,v}(\mathcal{O})$. This ends the proof of the proposition. \blacksquare

We are now in position to prove Theorem 2.6

Proof of Theorem 2.6:

Observe that

$$(31) \implies 0 \leq \phi_t^\pi(Q) - \phi_t(Q) \leq \phi_{\pi,t}(Q) - \phi_t^\pi(Q) + \phi_t^\pi(Q) - \phi_t(Q) = \phi_{\pi,t}(Q) - \phi_t(Q)$$

This implies that

$$\|\phi_t^\pi(Q) - \phi_t(Q)\|_2 \leq \|\phi_{\pi,t}(Q) - \phi_t(Q)\|_2$$

By Corollary 2.8, there exist some $\delta > 0$ such that for any $\pi \in B(\delta)$ we have the uniform estimate

$$\sup_{t \geq 0} \sup_{Q \in \mathbb{S}_r^+} \|\Xi_\pi(\phi_{\pi,t+v}(Q))\|_2 \leq \chi_1(\delta) [\|A_\pi - A\|_2 + \|R_\pi - R\|_2 + \|S_\pi - S\|_2]$$

for constant finite constant $\chi_1(\delta)$ whose values only depend on δ . We have

$$(22) \implies \sup_{\pi \in B(\delta)} \sup_{0 \leq t \leq v} \|\phi_{\pi,t}(Q)\|_2 \leq \chi_2(\delta) (1 + \|Q\|_2)$$

for some constant $\chi_2(\delta)$ whose values only depend on δ . Combining (23) with (34) we have

$$\begin{aligned} \|\phi_{\pi,t}(Q) - \phi_t(Q)\|_2 &\leq [\kappa_E(\chi_2(\delta) (1 + \|Q\|_2))]^2 \\ &\times [2\chi_2(\delta) (1 + \|Q\|_2) \|A_\pi - A\|_2 + \|R_\pi - R\|_2 + \chi_2(\delta)^2 (1 + \|Q\|_2)^2 \|S_\pi - S\|_2] \\ &\times [e^{-4\nu(t-v)} - e^{-4t\nu}]/(4\nu) \\ &+ \chi_1(\delta) \kappa_E(\chi_1(\delta))^2 [\|A_\pi - A\|_2 + \|R_\pi - R\|_2 + \|S_\pi - S\|_2] [1 - e^{-4\nu(t-v)}]/(4\nu) \end{aligned}$$

This ends the proof of the first assertion. To check the last assertion we simply let $t \uparrow \infty$. More precisely, observe that

$$\|P_\pi - P\|_2 \leq \|\phi_{\pi,t}(P_\pi) - \phi_t(P_\pi)\|_2 + \|\phi_t(P_\pi) - \phi_t(P)\|_2$$

This implies that

$$\begin{aligned} \|P_\pi - P\|_2 &\leq \kappa_\phi(\|P_\pi\|_2, \|P\|_2) e^{-2\nu t} \|P_\pi - P\|_2 \\ &+ [\chi_1(\delta) + e^{-4t\nu} \chi_2(\delta, \|P_\pi\|_2)] [\|A_\pi - A\|_2 + \|R_\pi - R\|_2 + \|S_\pi - S\|_2] \end{aligned}$$

Letting $t \rightarrow \infty$ we end the proof of the desired estimate. This ends the proof of the theorem. \blacksquare

2.3 Projection-type models

We consider projection-type mappings in (8) of the second type in (9). Let π be some positive map from \mathbb{M}_r into itself; that is $\pi(\mathbb{S}_r^+) \subseteq \mathbb{S}_r^+$. We first assume the matrices (A, R, S) are chosen so that

$$(\pi(A), \pi(A'), \pi(S), \pi(R)) = (A, A', R, S) \quad \text{and we let } \mathcal{B} \subset \mathbb{M}_r \text{ be a given matrix ring}$$

Also assume that the pair (π, \mathcal{B}) satisfies the following orthogonality property:

$$(H)_3 \quad \forall Q \in \mathbb{M}_r \quad \forall B \in \mathcal{B} \quad \pi(B[Q - \pi(Q)] + [Q - \pi(Q)]B) = 0 \quad (39)$$

In this situation, we have

$$\begin{aligned} \pi[(Q - B)(Q - B)'] &= \pi[(Q - \pi(Q))(Q - \pi(Q))'] + \pi(BB') \\ &\geq \pi[(Q - \pi(Q))(Q - \pi(Q))'] \geq 0 \end{aligned}$$

This shows that π can be interpreted as a π -orthogonal projection

$$(H)_3 \iff \pi(Q) = \operatorname{argmin}_{B \in \mathcal{B}} \pi[(Q - B)(Q - B)']$$

In addition, we have the Cauchy-Schwartz inequality

$$\pi[(Q - \pi(Q))(Q - \pi(Q))'] \geq 0 \implies \pi(QQ') \geq \pi(Q)\pi(Q')$$

Whenever \mathcal{B} is closed by transposition we have

$$\pi(Q') = \pi(Q)' \implies \pi(Q)\pi(Q)' \leq \pi(QQ')$$

The identity mapping $\pi = id$ and the set $\mathcal{B} = \mathbb{M}_r$ clearly satisfies the above properties.

The prototype of a non-trivial pair (\mathcal{B}, π) satisfying $(H)_3$ are orthogonal projections $\pi = \operatorname{proj}_{\mathcal{B}}$ (w.r.t. the Frobenius norm) onto cellular (a.k.a. coherent) algebras; that is a sub-algebra of matrices which are additionally closed under the Hadamard-Schur product and contains the identity elements Id and J , where J stands for the matrix with all ones entries. Up to a unitary change of basis, these projections can be reformulated in terms of block-diagonal matrices [45]. By [47, page 57], a sub-algebra of \mathbb{S}_r is a Bose-Mesner algebra [46] of some association scheme if and only if it contains I and J , and it is closed under the Hadamard-Schur product. This shows that cellular sub-algebras of \mathbb{S}_r coincide with the Bose-Mesner algebras (of some association scheme). We refer to Section 4.4 for a detailed discussion on these models.

The set $\mathcal{B} = \mathcal{M}_{r[1]} \oplus \dots \oplus \mathcal{M}_{r[n]} \subset \mathbb{M}_r$ (with $r = \sum_{1 \leq q \leq n} r[q]$) of block-diagonal matrices with null entries outside the blocks is also a matrix ring which is closed under the Hadamard-Schur product w.r.t. any matrix in \mathbb{M}_r ; but \mathcal{B} is not a cellular algebra since $J \notin \mathcal{B}$. The orthogonal projection from \mathbb{M}_r onto this \mathcal{B} is given by

$$\pi(Q) := L \odot Q \quad \text{with the block-diagonal matrix } L := \operatorname{diag}(J_1, \dots, J_n) \geq 0 \quad (40)$$

In the above display, J_i stands for the i -th block unit matrix w.r.t. the Hadamard-Schur product; that is the $(r[i] \times r[i])$ -square matrix with all unit entries. It is readily checked that (\mathcal{B}, π) satisfies condition $(H)_3$. We refer to Section 4.3 for a discussion on these models.

We let ϕ_t^π be the π -Riccati semigroup defined in Section 1.3. By (33) we have the domination property

$$\forall Q \in \mathbb{S}_r, \quad \phi_t^\pi(Q) \geq \phi_t(Q) \quad (41)$$

In contrast with the second order approximation models discussed in Section 2.2.1 these projection techniques don't depend on some perturbation index that quantifies the distance between π and the identity mapping.

When $(\pi(A), \pi(S), \pi(R)) \neq (A, R, S)$ we can replace (A, R, S) by their projections (A_π, R_π, S_π) . In this case, $\phi_{\pi,t}$ is the Riccati semigroup associated with the drift function $\text{Ricc}_\pi(Q)$ defined by $\text{Ricc}(Q)$ with (A, R, S) replaced by (A_π, R_π, S_π) . The difference between $\phi_{\pi,t}$ and ϕ_t can be analyzed as in Theorem 2.6. It is not possible to ensure that $\phi_{\pi,t}$ is arbitrarily close to ϕ_t without some continuity conditions.

Section 4.5 discusses a way to combine these projection-type models with the perturbation-type models discussed in Section 2.2.1

In the latter development of Section 2.3.1 we will provide exponential concentration inequalities that ensure the π -projected Riccati flows converge exponentially fast to the solution of the (nominal, Kalman-Bucy) Riccati equation, viz [1, 2], as the time horizon tends to ∞ , and as soon as condition $(H)_3$ is met. Speaking somehow loosely we shall show that

$$\phi_t \circ \pi = \phi_t^\pi \circ \pi \quad \text{and} \quad \phi_t^\pi \simeq \phi_t$$

Definition 2.11. We let $\varphi_t^\pi(Q)$ be the flow of the projected Riccati equation

$$\partial_t \varphi_t^\pi(Q) = \pi [\text{Ricc}(\varphi_t^\pi(Q))]$$

The next theorem shows that the flows φ_t^π and ϕ_t^π coincide with the π -projection of the Riccati flow $\pi \circ \phi_t$ as soon as we start from an initial state Q that satisfies $\pi(Q) = Q$ and $(A, R, S) \in \mathcal{B}^3$. It also provides an explicit description of the flow ϕ_t^π in terms of ϕ_t and π when $(H)_3$ is satisfied.

Theorem 2.12. Assume $(H)_3$. For any time horizon $t \geq 0$ we have the formula

$$\pi \circ \text{Ricc} \circ \pi = \text{Ricc}^\pi \circ \pi = \text{Ricc} \circ \pi = \pi \circ \text{Ricc}^\pi \circ \pi$$

as well as the semigroup commutation properties

$$\pi \circ \phi_t \circ \pi = \pi \circ \varphi_t^\pi \circ \pi = \varphi_t^\pi \circ \pi = \phi_t \circ \pi = \phi_t^\pi \circ \pi \quad (42)$$

In addition, we have the formula

$$\phi_t(Q) \leq \phi_t^\pi(Q) = [\phi_t \circ \pi](Q) + E_t(\pi(Q))(Q - \pi(Q))E_t(\pi(Q))' \quad (43)$$

Proof. Recall that $(A, A', R, S) \in \mathcal{B}$. Since \mathcal{B} is a matrix ring we have

$$\pi [A\pi(Q) + \pi(Q)A' + R - \pi(Q)S\pi(Q)] = A\pi(Q) + \pi(Q)A' + R - \pi(Q)S\pi(Q)$$

or equivalently

$$\pi \circ \text{Ricc} \circ \pi = \text{Ricc} \circ \pi$$

Also observe that

$$\begin{aligned} \text{Ricc}^\pi(Q) &= (A - \pi(Q)S)Q + Q(A - \pi(Q)S)' + R + \pi(Q)S\pi(Q) \\ &= \text{Ricc}(\pi(Q)) + (A - \pi(Q)S)(Q - \pi(Q)) + (Q - \pi(Q))(A - \pi(Q)S)' \\ &= [\pi \circ \text{Ricc} \circ \pi](Q) + (A - \pi(Q)S)(Q - \pi(Q)) + (Q - \pi(Q))(A - \pi(Q)S)' \end{aligned}$$

and thus

$$\text{Ricc}^\pi \circ \pi = \pi \circ \text{Ricc} \circ \pi = \pi \circ \text{Ricc}^\pi \circ \pi$$

Now, we also have

$$\pi^2 := \pi \circ \pi = \pi \Rightarrow \partial_t \pi(\varphi_t^\pi(Q)) = \partial_t \varphi_t^\pi(Q) \Rightarrow \pi(\varphi_t^\pi(Q)) = \varphi_t^\pi(Q) + \pi(Q) - Q$$

This implies that

$$\pi \circ \varphi_t^\pi \circ \pi = \varphi_t^\pi \circ \pi$$

This yields

$$\begin{aligned} \partial_t [\varphi_t^\pi \circ \pi](Q) &= \partial_t [\pi \circ \varphi_t^\pi \circ \pi](Q) = \pi([\text{Ricc} \circ \varphi_t^\pi \circ \pi](Q)) \\ &= \pi([\text{Ricc} \circ \pi \circ \varphi_t^\pi \circ \pi](Q)) = [\text{Ricc} \circ \pi \circ \varphi_t^\pi \circ \pi](Q) \\ &= [\text{Ricc} \circ \varphi_t^\pi \circ \pi](Q) \end{aligned}$$

and by the uniqueness of the solution of the Riccati equation we conclude that

$$\varphi_t^\pi \circ \pi = \phi_t \circ \pi \Rightarrow \pi \circ \phi_t \circ \pi = \pi \circ \varphi_t^\pi \circ \pi = \varphi_t^\pi \circ \pi = \phi_t \circ \pi$$

We also have

$$\begin{aligned} \partial_t [\varphi_t^\pi \circ \pi](Q) &= \partial_t [\pi \circ \varphi_t^\pi \circ \pi](Q) = \pi[\text{Ricc}^\pi([\pi \circ \varphi_t^\pi \circ \pi](Q))] \\ &= \text{Ricc}^\pi([\pi \circ \varphi_t^\pi \circ \pi](Q)) = \text{Ricc}^\pi([\varphi_t^\pi \circ \pi](Q)) \end{aligned}$$

which implies that

$$\phi_t^\pi \circ \pi = \varphi_t^\pi \circ \pi$$

This completes the proof of (42).

Now, we have

$$\begin{aligned} \partial_t [\phi_t^\pi(Q) - \phi_t(\pi(Q))] &= \text{Ricc}(\pi(\phi_t^\pi(Q))) - \text{Ricc}(\phi_t(\pi(Q))) \\ &\quad + (A - \phi_t(\pi(Q))S)(\phi_t^\pi(Q) - \pi(\phi_t^\pi(Q))) + (\phi_t^\pi(Q) - \pi(\phi_t^\pi(Q)))(A - \phi_t(\pi(Q))S)' \\ &= (A - \phi_t(\pi(Q))S)(\phi_t^\pi(Q) - \pi(\phi_t^\pi(Q))) + (\phi_t^\pi(Q) - \pi(\phi_t^\pi(Q)))(A - \phi_t(\pi(Q))S)' \end{aligned}$$

This implies that

$$\phi_t^\pi(Q) - \phi_t(\pi(Q)) = E_t(\pi(Q))(Q - \pi(Q))E_t(\pi(Q))'$$

The l.h.s. estimate in (43) is a consequence of the domination property (41). This ends the proof of the theorem. \blacksquare

2.3.1 Exponential contraction inequalities

We continue with the projection-type models.

Theorem 2.13. *For any $Q_1, Q_2 \in \mathbb{S}_r^+$ and for any $t \geq 0$ we have*

$$\|\phi_t^\pi(Q_2) - \phi_t^\pi(Q_1)\|_2 \leq \kappa_{\phi^\pi}(Q_1, Q_2) e^{-2\nu t} [\|\pi(Q_2) - \pi(Q_1)\|_2 + e^{-2\nu t} \|Q_2 - Q_1\|_2] \quad (44)$$

some finite constant $\kappa_{\phi^\pi}(Q_1, Q_2) < \infty$ whose values only depend on $(\|Q_1\|_2, \|Q_2\|_2)$. This implies the existence of an unique fixed point $P^\pi = \phi_t^\pi(P^\pi)$ with $\pi(P^\pi) = P$. In addition, for any $Q \in \mathbb{S}_r^+$ and for any $t \geq 0$ we have

$$\|\pi[\phi_t^\pi(Q)] - \phi_t^\pi(Q)\|_2 \leq \kappa_{\phi^\pi}(Q, \pi(Q)) e^{-4\nu t} \|\pi(Q) - Q\|_2 \quad (45)$$

Proof. We have

$$\begin{aligned}\phi_t^\pi(Q) &= \phi_t^\pi(\pi(Q)) + E_t(\pi(Q))(Q - \pi(Q))E_t(\pi(Q))' \\ &= \phi_t(\pi(Q)) + E_t(\pi(Q))(Q - \pi(Q))E_t(\pi(Q))'\end{aligned}$$

This implies that

$$\begin{aligned}\phi_t^\pi(Q_1) - \phi_t^\pi(Q_2) &= \phi_t(\pi(Q_1)) - \phi_t(\pi(Q_2)) \\ &\quad + [E_t(\pi(Q_2))(\pi(Q_2) - Q_2)E_t(\pi(Q_2))' - E_t(\pi(Q_1))(\pi(Q_1) - Q_1)E_t(\pi(Q_1))']\end{aligned}$$

Using (23) we find that

$$\|\phi_t(\pi(Q_1)) - \phi_t(\pi(Q_2))\|_2 \leq \kappa_\phi(\pi(Q_2), \pi(Q_1)) e^{-2\nu t} \|\pi(Q_2) - \pi(Q_1)\|_2$$

To estimate the second term, we use the decomposition

$$\begin{aligned}E_t(\pi(Q_2)) (\pi(Q_2) - Q_2) E_t(\pi(Q_2))' - E_t(\pi(Q_1)) (\pi(Q_1) - Q_1) E_t(\pi(Q_1))' \\ = [E_t(\pi(Q_2)) - E_t(\pi(Q_1))] (\pi(Q_2) - Q_2) E_t(\pi(Q_2))' \\ + E_t(\pi(Q_1)) [\{(\pi(Q_2) - \pi(Q_1)) - (Q_2 - Q_1)\} E_t(\pi(Q_2))'] \\ + E_t(\pi(Q_1)) [(\pi(Q_1) - Q_1) [E_t(\pi(Q_2))' - E_t(\pi(Q_1))']]\end{aligned}$$

Combining (23) with (25) we find that

$$\begin{aligned}\|E_t(\pi(Q_2)) (\pi(Q_2) - Q_2) E_t(\pi(Q_2))' - E_t(\pi(Q_1)) (\pi(Q_1) - Q_1) E_t(\pi(Q_1))'\|_2 \\ \leq \kappa_E(\pi(Q_2), \pi(Q_1)) e^{-3\nu t} \\ \times \|\pi(Q_2) - \pi(Q_1)\|_2 [\kappa_E(\pi(Q_2))\|\pi(Q_2) - Q_2\|_2 + \kappa_E(\pi(Q_1))\|\pi(Q_1) - Q_1\|_2] \\ + \kappa_E(Q_1)\kappa_E(Q_2) e^{-4\nu t} [\|\pi(Q_2) - \pi(Q_1)\|_2 + \|Q_2 - Q_1\|_2]\end{aligned}$$

To prove (45) we recall from Theorem 2.12 that $\pi \circ \phi_t^\pi = \phi_t^\pi \circ \pi$. This implies that

$$\begin{aligned}\|\pi [\phi_t^\pi(Q_2)] - \phi_t^\pi(Q_1)\|_2 \\ \leq \kappa_{\phi^\pi}(Q_1, \pi(Q_2)) e^{-2\nu t} [\|\pi(Q_2) - \pi(Q_1)\|_2 + e^{-2\nu t} \|\pi(Q_2) - Q_1\|_2]\end{aligned}$$

If we set $Q_1 = Q_2$ we obtain (45). This ends the proof of the theorem. \blacksquare

Combining (44) with the fact that $\phi_t^\pi \circ \pi = \phi_t \circ \pi$ and $\pi(P^\pi) = P$ we readily prove the following estimate.

Corollary 2.14. *For any $Q \in \mathbb{S}_r^+$ and for any $t \geq 0$ we have*

$$\|\phi_t^\pi(Q) - \phi_t(\pi(Q))\|_2 \leq \kappa_{\phi^\pi}(Q, \pi(Q)) e^{-4\nu t} \|Q - \pi(Q)\|_2 \implies P^\pi = P \quad (46)$$

In addition, we have

$$\|\phi_t^\pi(Q) - \phi_t(Q)\|_2 \leq e^{-2\nu t} [\kappa_\phi(Q, \pi(Q)) + \kappa_{\phi^\pi}(Q, \pi(Q)) e^{-2\nu t}] \|Q - \pi(Q)\|_2 \quad (47)$$

The estimate (47) is a direct consequence of (24) and (46). Replacing Q by $P = P^\pi$ in (46) we obtain the following exponential decays to equilibrium.

Corollary 2.15. *For any $Q \in \mathbb{S}_r^+$ and for any $t \geq 0$ we have the local Lipschitz estimate*

$$\|\phi_t^\pi(Q) - P\|_2 \leq \kappa_{\phi^\pi}(Q, P) e^{-2\nu t} [\|\pi(Q) - P\|_2 + e^{-2\nu t} \|Q - P\|_2] \quad (48)$$

This yields the uniform estimate

$$\|\phi_t^\pi(Q)\|_2 := \sup_{t \geq 0} \|\phi_t^\pi(Q)\|_2 \leq \|P\|_2 + \kappa_{\phi^\pi}(Q, P) [\|\pi(Q) - P\|_2 + \|Q - P\|_2]$$

Combining (25) with Corollary 2.15 we prove the following local Lipschitz contraction.

Corollary 2.16. *For any $0 \leq s \leq t$ and any $Q \in \mathbb{S}_r^+$*

$$\|E_{t|s}(\phi_s^\pi(Q)) - E_{t|s}(P)\|_2 \leq \kappa_{E,\pi}(Q, P) e^{-\nu(t+s)} [\|\pi(Q) - P\|_2 + e^{-2\nu s} \|Q - P\|_2]$$

some finite constant $\kappa_{E,\pi}(Q, P) < \infty$ whose values only depend respectively on $(\|P\|_2, \|Q\|_2)$.

3 Kalman-Bucy stochastic flows

3.1 Perturbation-type models

We consider the projection models discussed in Section 2.2. We set

$$\sigma_\delta^2(Q) := 2\sqrt{2} \kappa_{\delta,E}(Q) \left[r \left[\|R\|_2 + \|S\|_2 (\delta + \|\phi(Q)\|_{\delta,2})^2 \right] / ((1-\delta)\nu) \right]^{1/2}$$

with

$$\kappa_{\delta,E}(Q) := \kappa_E(\|Q\|_2) \exp[\chi_2(\delta, \|Q\|_2)/(4\nu)] \quad \text{and} \quad \|\phi(Q)\|_{\delta,2} := \sup_{t \geq 0} \sup_{\pi \in B(\delta)} \|\phi_t^\pi(Q)\|_2 < \infty$$

where $\chi_2(\cdot, \cdot) < \infty$ is introduced in Theorem 2.6 and $\nu > 0$ and $\kappa_E(\cdot) < \infty$ are defined in (23).

Recall the semigroup and stochastic flow notation defined in Section 1.3. The first lemma in this section concerns the convergence of the perturbed Kalman-Bucy filter to the true underlying signal process, both in a mean-square sense and in terms of actual sample paths.

Lemma 3.1. *Assume $(H)_0$ and $(H)_2$ are satisfied. For any $\epsilon > 0$ there exists some parameter $0 < \delta < \epsilon$ such that for any $\pi \in B(\delta)$, $0 \leq s \leq t$, $Q \in \mathbb{S}_r^+$ and any $n \geq 1$ we have*

$$\mathbb{E} [\|\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s)\|_2^{2n} | X_s]^{1/2n} \leq \kappa_{\delta,E}(Q) e^{-2(1-\epsilon)\nu(t-s)} \|X_s - x\|_2 + \sqrt{n} \sigma_\delta(Q)$$

In addition, the conditional probability of the following event

$$\|\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s)\|_2 \leq \kappa_{\delta,E}(Q) e^{-2(1-\epsilon)\nu(t-s)} \|X_s - x\|_2 + \sigma_\delta(Q) \frac{e^2}{\sqrt{2}} \left[\frac{1}{2} + (z + \sqrt{z}) \right]$$

given the state variable X_s is greater than $1 - e^{-z}$, for any $z \geq 0$.

Proof. We have

$$d [\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s)] = [A - \pi(\phi_{s,t}^\pi(Q)) S] [\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s)] dt + dM_{s,t}^\pi$$

In the above display, $t \in [s, \infty[\mapsto M_{s,t}^\pi$ stands for the r -dimensional martingale

$$dM_{s,t}^\pi := R^{1/2} dW_t + \pi(\phi_{s,t}^\pi(Q)) C' R_2^{-1/2} dV_t$$

with angle bracket

$$(\partial_t \langle M_{s,t}^\pi(k), M_{s,t}^\pi(l) \rangle)_{1 \leq k, l \leq r} := R + \pi(\phi_{s,t}^\pi(Q)) S \pi(\phi_{s,t}^\pi(Q))$$

This yields the formula

$$\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s) - E_{t|s}^\pi(Q)(x - X_s) = \int_s^t E_{u,t|s}^\pi(Q) dM_{s,u}^\pi$$

with the exponential semigroup $E_{u,t|s}^\pi(Q)$ defined for any $s \leq u \leq t$ by

$$E_{u,t|s}^\pi(Q) = \exp \left(\int_u^t [A - \pi(\phi_{s,u}^\pi(Q)) S] du \right)$$

We have the decomposition

$$A - \pi(\phi_{s,u}^\pi(Q)) S = A - \phi_{s,u}(Q) S + [\phi_{s,u}(Q) - \phi_{s,u}^\pi(Q) + \phi_{s,u}^\pi(Q) - \pi(\phi_{s,u}^\pi(Q))] S$$

By (23) we have

$$\|E_{t|s}(Q_1)\|_2 \leq \kappa_E(\|Q_1\|_2) e^{-2\nu(t-s)}$$

By Theorem 2.6 there exists some $\delta > 0$ such that for any $\pi \in B(\delta)$

$$\|\pi[\phi_t^\pi(Q)] - \phi_t^\pi(Q)\|_2 \leq \delta \implies \sup_{t \geq 0} \sup_{\pi \in B(\delta)} \|\pi[\phi_t^\pi(Q)]\|_2 \leq \delta + \|\phi(Q)\|_{\delta,2}$$

In addition, we have

$$\|\phi_{s,t}(Q) - \phi_{s,u}^\pi(Q)\|_2 \leq [\chi_1(\delta) + e^{-4(t-s)\nu} \chi_2(\delta, \|Q\|_2)] \gamma(\pi)$$

Applying Lemma 1.5 we find that

$$\|E_{u,t|s}^\pi(Q)\|_2 \leq \kappa_{\delta,E}(Q) \exp[-\{2\nu - \delta - \gamma(\pi) \kappa_E(\|Q\|_2) \chi_1(\delta)\} (t-s)]$$

For any $\epsilon > 0$ we choose $\epsilon > \delta > \delta' > 0$ and so that for any $\pi \in B(\delta')$

$$\delta + \gamma(\pi) \kappa_E(\|Q\|_2) \chi_1(\delta) \leq 2\epsilon\nu \implies \|E_{u,t|s}^\pi(Q)\|_2 \leq \kappa_{\delta,E}(Q) \exp[-2(1-\epsilon)\nu(t-s)]$$

Following the proof of Lemma 5.2 in [2], for any $n \geq 1$ we have

$$\begin{aligned} & \mathbb{E} \left[\left(\left\| \int_s^t E_{u,t|s}^\pi(Q) dM_{s,u}^\pi \right\|_2^{2n} \right) \right]^{\frac{1}{n}} \\ & \leq 4^2 n^r \int_s^t \|R + \pi(\phi_{s,u}^\pi(Q)) S \pi(\phi_{s,u}^\pi(Q))\|_2 \|E_{u,t|s}^\pi(Q)\|_2 du \\ & \leq 8n^r \kappa_{\delta,E}(Q) \left[\|R\|_2 + \|S\|_2 (\delta + \|\phi(Q)\|_{\delta,2})^2 \right] / ((1-\epsilon)\nu) \leq n \sigma_\delta^2(Q) \end{aligned}$$

The end of the proof of the first assertion is now easily completed. The proof of the exponential concentration inequality follows the same line of argument as the proof of Proposition 5.1 in [2]. This ends the proof of the lemma. \blacksquare

The next three theorems concern convergence of the π -perturbed Kalman-Bucy filter/diffusion to the true, optimal, Kalman-Bucy filter [2]. The first concerns the stochastic flow of the two filters themselves, while the second two theorems concern the associated (conditional) distributions.

Theorem 3.2. *Assume $(H)_0$ and $(H)_2$ are satisfied. In this situation, there exists some parameter $\delta > 0$ such that for any $0 < \epsilon < \delta$, $\pi \in B(\epsilon)$, $0 \leq s \leq t$, $Q \in \mathbb{S}_r^+$, and any $n \geq 1$ we have*

$$\mathbb{E} [\|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^{2n} \mid X_s] \leq \epsilon \chi(\delta, Q) \left[\sqrt{n} + e^{-(1-\epsilon)\nu(t-s)} \|X_s - x\|_2 \right]$$

for some finite constants $\chi(\delta, Q)$ whose value only depends on the parameters $(\delta, \|Q\|_2)$.

Proof. We have

$$\begin{aligned} d[\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)] &= \{[A - \pi(\phi_{s,t}^\pi(Q))S] - [A - \phi_{s,t}(Q)S]\} \psi_{s,t}^\pi(x, Q) \\ &\quad - [A - \phi_{s,t}(Q)S] [\psi_{s,t}(x, Q) - \psi_{s,t}^\pi(x, Q)] dt + [\pi(\phi_{s,t}^\pi(Q)) - \phi_{s,t}(Q)] C'\Sigma^{-1} dY_t \end{aligned}$$

This implies that

$$\begin{aligned} d[\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)] &= [\phi_{s,t}(Q) - \pi(\phi_{s,t}^\pi(Q))] S \psi_{s,t}^\pi(x, Q) dt \\ &\quad + [\pi(\phi_{s,t}^\pi(Q)) - \phi_{s,t}(Q)] C'\Sigma^{-1} (C\theta_{s,t}(X_s)dt + R_2^{1/2}dV_t) \\ &\quad + [A - \phi_{s,t}(Q)S] [\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)] dt \\ &= [A - \phi_{s,t}(Q)S] [\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)] dt \\ &\quad + [\phi_{s,t}(Q) - \pi(\phi_{s,t}^\pi(Q))] S [\psi_{s,t}^\pi(x, Q) - \theta_{s,t}(X_s)] dt + dM_{s,t}^\pi \end{aligned}$$

with the r -dimensional martingale $t \in [s, \infty[\mapsto M_{s,t}^\pi$ defined by

$$dM_{s,t}^\pi = [\pi(\phi_{s,t}^\pi(Q)) - \phi_{s,t}(Q)] C'R_2^{-1/2}dV_t$$

This implies that

$$\begin{aligned} \psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q) &= \int_s^t E_{u,t|s}(Q) [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)] du + \int_s^t E_{u,t|s}(Q) dM_{s,u}^\pi \end{aligned}$$

Arguing as in the proof of Lemma 3.1, there exists some $0 < \epsilon < \delta$ such that for any $\pi \in B(\epsilon)$

$$\|\pi[\phi_t^\pi(Q)] - \phi_t^\pi(Q)\|_2 \leq \epsilon$$

and

$$\|\phi_{s,u}(Q) - \phi_{s,u}^\pi(Q)\|_2 \leq \epsilon \left[\chi_1(\delta) + e^{-4(u-s)\nu} \chi_2(\delta, \|Q\|_2) \right]$$

By the generalized Minkowski inequality, we have

$$\begin{aligned} & \left\| \int_s^t E_{u,t|s}(Q) [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)] du \right\|_2 \leq \epsilon \kappa_E(Q) \|S\|_2 \\ & \quad \times \int_s^t \left[e^{-(t-u)\nu} (1 + \chi_1(\delta)) + e^{-4(t-s)\nu} \chi_2(\delta, \|Q\|_2) \right] \|\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)\|_2 du \end{aligned}$$

This implies that

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_s^t E_{u,t|s}(Q) [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)] du \right\|_2^{2n} \mid X_s \right]^{\frac{1}{2n}} \\ & \leq \epsilon \kappa_E(Q) \|S\|_2 \int_s^t \left[e^{-(t-u)\nu} (1 + \chi_1(\delta)) + e^{-4(t-s)\nu} \chi_2(\delta, \|Q\|_2) \right] \\ & \quad \times \mathbb{E} [\|\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)\|_2^{2n} \mid X_s]^{\frac{1}{2n}} du \end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_s^t E_{u,t|s}(Q) [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)] du \right\|_2^{2n} \mid X_s \right]^{\frac{1}{2n}} \\ & \leq \epsilon \kappa_E(Q) \|S\|_2 \int_s^t \left[e^{-(t-u)\nu} (1 + \chi_1(\delta)) + e^{-4(t-s)\nu} \chi_2(\delta, \|Q\|_2) \right] \\ & \quad \times \left[\kappa_{\delta,E}(Q) e^{-2(1-\epsilon)\nu(u-s)} \|X_s - x\|_2 + \sqrt{n} \sigma_\delta(Q) \right] du \end{aligned}$$

This yields the estimate

$$\begin{aligned} & \mathbb{E} \left[\left\| \int_s^t E_{u,t|s}(Q) [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\psi_{s,u}^\pi(x, Q) - \theta_{s,u}(X_s)] du \right\|_2^{2n} \mid X_s \right]^{\frac{1}{2n}} \\ & \leq \epsilon \kappa_E(Q) \|S\|_2 \left\{ \sqrt{n} \left[\bar{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \bar{\chi}_2(\delta, Q) \right] \right. \\ & \quad \left. + \|X_s - x\|_2 e^{-(1-\epsilon)\nu(t-s)} \left[\underline{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \underline{\chi}_2(\delta, Q) \right] \right\} \end{aligned}$$

with

$$\bar{\chi}_1(\delta, Q) := \sigma_\delta(Q) (1 + \chi_1(\delta))/\nu \quad \text{and} \quad \bar{\chi}_2(\delta, Q) := \sigma_\delta(Q) \chi_2(\delta, \|Q\|_2)$$

and

$$\underline{\chi}_1(\delta, Q) := \kappa_{\delta,E}(Q) (1 + \chi_1(\delta))/((1-\delta)\nu) \quad \text{and} \quad \underline{\chi}_2(\delta, Q) := \kappa_{\delta,E}(Q) \chi_2(\delta, \|Q\|_2)/(2(1-\delta)\nu)$$

Following the proof of Lemma 5.2 in [2], for any $n \geq 1$ we have

$$\begin{aligned} & \mathbb{E} \left[\left(\left\| \int_s^t E_{u,t|s}(Q) dM_{s,u}^\pi \right\|_2^{2n} \right) \right]^{\frac{1}{n}} \\ & \leq 4^2 n r \int_s^t \left\| [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] S [\phi_{s,u}(Q) - \pi(\phi_{s,u}^\pi(Q))] \right\|_2 \|E_{u,t|s}(Q)\|_2 du \\ & \leq 8 \epsilon^2 n r \kappa_E(Q) \|S\|_2 / \nu = \epsilon^2 \bar{\sigma}^2(Q) \end{aligned}$$

with

$$\bar{\sigma}^2(Q) := 8 r n \kappa_E(Q) \|S\|_2 / \nu$$

This yields

$$\begin{aligned} & \epsilon^{-1} \mathbb{E} \left[\|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^{2n} \right]^{\frac{1}{2n}} \\ & \leq \sqrt{n} \bar{\sigma}(Q) + \kappa_E(Q) \|S\|_2 \left\{ \sqrt{n} [\bar{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \bar{\chi}_2(\delta, Q)] \right. \\ & \quad \left. + \|X_s - x\|_2 e^{-(1-\epsilon)\nu(t-s)} [\underline{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \underline{\chi}_2(\delta, Q)] \right\} \\ & \leq \sqrt{n} [\bar{\sigma}(Q) + \kappa_E(Q) \|S\|_2 [\bar{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \bar{\chi}_2(\delta, Q)]] + \\ & \quad + e^{-(1-\epsilon)\nu(t-s)} \|X_s - x\|_2 [\underline{\chi}_1(\delta, Q) + e^{-3(t-s)\nu} \underline{\chi}_2(\delta, Q)] \end{aligned}$$

This ends the proof of the theorem. ■

Theorem 3.3. Assume $(H)_0$ and $(H)_2$ are satisfied. In this situation, for any $s + v \leq t$, and any $Q \in \mathbb{S}_r^+$ we have

$$\begin{aligned} & \text{Ent}(\eta_{s,t}^\pi(x, Q) \mid \eta_{s,t}(x, Q)) \\ & \leq \frac{1}{2} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \left[\|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \frac{5}{2} \sqrt{r} \|\phi_{s,t}(Q) - \phi_{s,t}^\pi(Q)\|_2 \right] \end{aligned}$$

Proof. The Boltzmann relative entropy of $\eta_{s,t}^\pi(x, Q)$ w.r.t. $\eta_{s,t}(x, Q)$ is given by the formula

$$\begin{aligned} \text{Ent}(\eta_{s,t}^\pi(x, Q) \mid \eta_{s,t}(x, Q)) &= -\frac{1}{2} [\text{tr}(I - \phi_{s,t}(Q)^{-1} \phi_{s,t}^\pi(Q)) + \log \det(\phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{-1})] \\ & \quad + \frac{1}{2} \langle (\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)), \phi_{s,t}(Q)^{-1} (\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)) \rangle \end{aligned}$$

By Corollary 2.8, for any $t \geq s + v$ we have

$$\begin{aligned} \text{Ent}(\eta_{s,t}^\pi(x, Q) \mid \eta_{s,t}(x, Q)) &= -\frac{1}{2} [\text{tr}(I - \phi_{s,t}(Q)^{-1} \phi_{s,t}^\pi(Q)) + \log \det(\phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{-1})] \\ & \quad + \frac{1}{2} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 \end{aligned}$$

In addition, there exists some $\delta > 0$ s.t. for any $\pi \in B(\delta)$

$$\begin{aligned} \|I - \phi_{s,t}(Q)^{-1} \phi_{s,t}^\pi(Q)\|_2 &= \|(\phi_{s,t}(Q) - \phi_{s,t}^\pi(Q)) \phi_{s,t}(Q)^{-1}\|_2 \\ &\leq (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \|\phi_{s,t}(Q) - \phi_{s,t}^\pi(Q)\|_2 \leq \frac{1}{2\sqrt{r}} \end{aligned}$$

This implies that

$$\begin{aligned} \text{Ent}(\eta_{s,t}^\pi(x, Q) \mid \eta_{s,t}(x, Q)) &= \frac{1}{2} \text{tr}(\phi_{s,t}(Q)^{-1} [\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)]) \\ &\quad + \frac{1}{2} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \left[\|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \frac{3}{2} \sqrt{r} \|\phi_{s,t}(Q) - \phi_{s,t}^\pi(Q)\|_2 \right] \end{aligned}$$

The last assertion is a consequence of the following lemma applied to $A = I - \phi_{s,t}(Q)^{-1} \phi_{s,t}^\pi(Q)$.

Lemma 3.4. *For any $(r \times r)$ -matrix A we have*

$$\|A\|_2 < \frac{1}{2\sqrt{r}} \implies |\log \det(I - A)| \leq \frac{3}{2} \sqrt{r} \|A\|_2$$

Proof. For any $n \geq 1$ we have

$$|\text{tr}(A^n)| \leq \|A\|_F^n \leq \sqrt{r}^n \|A\|_2^n$$

Using the well-known trace formulae

$$\log \det(I - A) = \text{tr}(\log(I - A)) = - \sum_{n \geq 1} n^{-1} \text{tr}(A^n)$$

we conclude that

$$|\log \det(I - A)| \leq -\log(1 - \sqrt{r} \|A\|_2)$$

The last assertion comes from the inequality

$$0 \leq -\log(1 - u) \leq u + \frac{1}{2} \frac{u^2}{1 - u} = u \left(1 + \frac{1}{2} \frac{u}{1 - u} \right) \leq 3u/2$$

which is valid for any $u \in [0, 1/2[$. This ends the proof of the lemma. ■

To take the final step in the proof of the theorem we note that $\phi_{s,t}^\pi(Q) \geq \phi_{s,t}(Q)$ implies

$$\begin{aligned} \text{tr}(\phi_{s,t}(Q)^{-1} [\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)]) &\leq (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \text{tr}([\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)]) \\ &\leq \sqrt{r} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \|\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)\|_2 \end{aligned}$$

This ends the proof of the theorem. ■

Theorem 3.5. *Assume $(H)_0$ and $(H)_2$ are satisfied. For any $Q \in \mathbb{S}_r^+$, and for $t \geq s + v$ we have the almost sure Wasserstein estimate*

$$\begin{aligned} \mathbb{W}_2[\eta_{s,t}^\pi(x, Q), \eta_{s,t}(x, Q)]^2 &\leq \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \text{tr}[\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)] \\ &\quad + 4r [\varpi_+^c(\mathcal{C}) + 1/\varpi_-^o] [\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c] \|\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)\|_2 \end{aligned}$$

In addition, for any $n \geq 1$ and any $t \geq s + v$ we have

$$\begin{aligned} & \mathbb{W}_{2n} [\eta_{s,t}(x_1, Q_1), \eta_{s,t}^\pi(x_2, Q_2)] \\ & \leq \|\psi_{s,t}(x, Q) - \psi_{s,t}^\pi(x, Q)\| \\ & \quad + \sqrt{\frac{rn}{2}} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c)^{1/2} \|\phi_{t-s}(x_1, Q_1) - \phi_{t-s}^\pi(x_2, Q_2)\|_2 e^{1/2 + \frac{3}{4n}} \end{aligned}$$

Proof. The \mathbb{L}_2 -Wasserstein distance between the Gaussian distributions $\eta_{s,t}^\pi(x, Q)$, and $\eta_{s,t}(x, Q)$ is given by

$$\begin{aligned} & \mathbb{W}_2 [\eta_{s,t}^\pi(x, Q), \eta_{s,t}(x, Q)]^2 \\ & = \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \text{tr} \left[\phi_{s,t}(Q) + \phi_{s,t}^\pi(Q) - 2 [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} \right] \end{aligned}$$

A proof of this formula can be found in [37, 40]. We assume that

$$\left[\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2} \right]^{1/2} \geq 0$$

is the principal square root of the positive definite matrix $\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2} \geq 0$. Also observe that

$$\begin{aligned} \phi_{s,t}^\pi(Q) \geq \phi_{s,t}(Q) & \Rightarrow \phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2} \geq \phi_{s,t}(Q)^2 \\ & \geq \lambda_{\min}(\phi_{s,t}(Q))^2 Id \\ & \geq (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c)^{-2} Id \end{aligned}$$

as soon as $t \geq s + v$. The last estimate is a consequence of Theorem 2.4.

Observe that

$$\phi_{s,t}(Q) + \phi_{s,t}^\pi(Q) - 2 \left[\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2} \right]^{1/2} = \phi_{s,t}^\pi(Q) - \phi_{s,t}(Q) \geq 0$$

This implies that

$$\begin{aligned} & \mathbb{W}_2 [\eta_{s,t}^\pi(x, Q), \eta_{s,t}(x, Q)]^2 \\ & = \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \text{tr} [\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)] \\ & \quad + 2 \text{tr} \left[[\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} - [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} \right] \\ & \leq \|\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)\|_2^2 + \text{tr} [\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)] \\ & \quad + 2r \left\| [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} - [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} \right\|_2 \end{aligned}$$

Using (15) we have

$$\begin{aligned} & \left\| [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} - [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} \right\|_2 \\ & \leq 2 (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c) \|\phi_{s,t}(Q)^{1/2} [\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)] \phi_{s,t}(Q)^{1/2}\|_2 \\ & \leq 2 [\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c] \|\phi_{s,t}(Q)^{1/2}\|_2^2 \|\phi_{s,t}^\pi(Q) - \phi_{s,t}(Q)\|_2 \end{aligned}$$

By Corollary 2.8 we conclude that

$$\begin{aligned} & \| [\phi_{s,t}(Q)^{1/2} \phi_{s,t}(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} - [\phi_{s,t}(Q)^{1/2} \phi_{s,t}^\pi(Q) \phi_{s,t}(Q)^{1/2}]^{1/2} \|_2 \\ & \leq 2 [\varpi_+^c(\mathcal{O}) + 1/\varpi_-^o] [\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c] \| \phi_{s,t}^\pi(Q) - \phi_{s,t}(Q) \|_2 \end{aligned}$$

This ends the proof of the first assertion.

Observe that

$$\overline{\psi}_{s,t}(x, Q) \stackrel{law}{=} \psi_{s,t}(x, Q) + \phi_{t-s}(x, Q)^{1/2} Z$$

and

$$\overline{\psi}_{s,t}^\pi(x, Q) \stackrel{law}{=} \psi_{s,t}^\pi(x, Q) + \phi_{t-s}^\pi(x, Q)^{1/2} Z$$

where Z stands for an r -dimensional Gaussian random variable with unit covariance matrix, and $\phi_{t-s}(x, Q)^{1/2}$ stands for the principal square root of $\phi_{t-s}(x, Q)$. Combining (15) with Theorem 2.4 for any $n \geq 1$ and any $t \geq s + v$ we have

$$\begin{aligned} & \mathbb{W}_{2n} [\eta_{s,t}(x_1, Q_1), \eta_{s,t}^\pi(x_2, Q_2)] \\ & \leq \| \psi_{s,t}(x, Q) - \psi_{s,t}^\pi(x, Q) \| \\ & \quad + \sqrt{\frac{rn}{2}} (\varpi_+^o(\mathcal{C}) + 1/\varpi_-^c)^{1/2} \| \phi_{t-s}(x_1, Q_1) - \phi_{t-s}^\pi(x_2, Q_2) \|_2 e^{1/2 + \frac{3}{4n}} \end{aligned}$$

To check the last assertion, we use Stirling approximation to prove that

$$\mathbb{E} \left[\left\| \sum_{1 \leq k \leq r} Z_k^2 \right\|^n \right]^{\frac{1}{n}} \leq \sum_{1 \leq k \leq r} \mathbb{E} [Z_1^{2n}]^{\frac{1}{n}} = \frac{r}{2} \left[\frac{(2n)!}{n!} \right]^{\frac{1}{n}} \leq 2 r n e^{1 + \frac{3}{2n}}$$

This ends the proof of the theorem. ■

3.2 Projection-type models

We consider the projection models discussed in Section 2.3. The semigroup commutation properties (42) already imply that

$$\psi_{s,t}^\pi(x, \pi(Q)) = \psi_{s,t}(x, \pi(Q)) \quad \text{and} \quad \overline{\psi}_{s,t}^\pi(x, \pi(Q)) = \overline{\psi}_{s,t}(x, \pi(Q))$$

Since $\pi(P) = P = P_\pi = \pi(P_\pi)$ we the steady state Kalman-Bucy diffusions coincide; that is we have that

$$\psi_{s,t}^\pi(x, P_\pi) = \psi_{s,t}(x, P) \quad \text{and} \quad \overline{\psi}_{s,t}^\pi(x, P) = \overline{\psi}_{s,t}(x, P)$$

By theorem 2.12 we have

$$\begin{cases} d\psi_{s,t}^\pi(x, Q) &= [A - \pi(\phi_{s,t}^\pi(Q)) S] \psi_{s,t}^\pi(x, Q) dt + \pi(\phi_{s,t}^\pi(Q)) C' \Sigma^{-1} dY_t \\ \partial_t \pi(\phi_{s,t}^\pi(Q)) &= \text{Ric}(\pi(\phi_{s,t}^\pi(Q))) \end{cases}$$

This implies that

$$\psi_{s,t}^\pi(x, Q) = \psi_{s,t}(x, \pi(Q)) \quad \text{and} \quad \overline{\psi}_{s,t}^\pi(x, Q) = \overline{\psi}_{s,t}(x, \pi(Q))$$

Thus, we have the decompositions

$$\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q) = \psi_{s,t}(x, \pi(Q)) - \psi_{s,t}(x, Q)$$

and

$$\overline{\psi}_{s,t}^\pi(x, Q) - \overline{\psi}_{s,t}(x, Q) = \overline{\psi}_{s,t}(x, \pi(Q)) - \overline{\psi}_{s,t}(x, Q)$$

These formulae shows that the convergence analysis of $\psi_{s,t}^\pi(x, Q) - \psi_{s,t}(x, Q)$ and $\overline{\psi}_{s,t}^\pi(x, Q) - \overline{\psi}_{s,t}(x, Q)$ to 0, as the time horizon $(t - s) \uparrow \infty$, reduces exactly to the stability properties of the Kalman-Bucy diffusion discussed in the article [2]. We point to this detailed literature for the exact Kalman-Bucy convergence results.

4 Some applications

4.1 Variance inflation models

We let $\Pi := \{\pi_\epsilon : \epsilon \in [0, 1]\}$ be the set of mappings

$$\pi_\epsilon(Q) = Q + \epsilon T \implies \Gamma_{\pi_\epsilon}(Q) = \epsilon^2 TST$$

indexed by $\epsilon \in [0, 1]$ and a given reference matrix $T \geq 0$. In this situation, the δ -balls around the identity mapping are given for any $\delta \leq 1$ by the compact sets

$$B(\delta \|T\|_2) = \{\pi_\epsilon : \epsilon \in [0, \delta]\} \subset \Pi$$

Conditions $(H)_0$ and $(H)_1$ are clearly met with

$$\begin{array}{llll} B_0 & = & \epsilon^2 TST & B_1 & = & 0 & B_2 & = & 0 & \mathcal{R}(Q) & = & 0 \\ R_\pi & = & R + \epsilon^2 TST & A_\pi & = & A & S_\pi & = & S & \implies & \Xi_\pi(Q) & = & \epsilon^2 TST \end{array}$$

To check $(H)_2$ we observe that

$$\begin{aligned} R^{-1/2} R_\pi R^{-1/2} - Id &= \epsilon^2 R^{-1/2} TST R^{-1/2} \\ \implies R^{-1/2} R_\pi R^{-1/2} &\leq (1 + \epsilon^2 \|R^{-1/2} TST R^{-1/2}\|) Id \\ \implies R &\leq R_\pi \leq R (1 + \epsilon^2 \|R^{-1/2} TST R^{-1/2}\|) Id \implies (H)_2 \end{aligned}$$

In this situation Theorem 2.6 yields the following corollary.

Corollary 4.1. *There exists some $\delta \in [0, 1]$ such that for any $\epsilon \in [0, \delta]$ and for any time horizon $t \geq 0$ and any $Q \in \mathbb{S}_r^+$ we have*

$$\|\phi_t^{\pi_\epsilon}(Q) - \phi_t(Q)\|_2 \leq \epsilon^2 [\chi_1(\delta) + e^{-4t\nu} \chi_2(\delta, \|Q\|_2)]$$

for some finite constant $\chi_1(\delta)$, resp. $\chi_2(\delta, \|Q\|)$, whose values only depend on the parameter δ , resp. on $(\delta, \|Q\|)$. In addition, for any $\epsilon \in [0, \delta]$ we have

$$\|P_{\pi_\epsilon} - P\|_2 \leq \epsilon^2 \chi_1(\delta)$$

4.2 Mean repulsion models

As their name indicates, mean repulsion models are defined by adding an extra repulsion term around the sample averages in the nonlinear diffusion (4). Consider the nonlinear diffusion

$$\begin{aligned}
d\overline{X}_t &= \left[A \overline{X}_t dt - T_1(P_t)(\overline{X}_t - \hat{X}_t) \right] dt + R^{1/2} d\overline{W}_t \\
&\quad + P_t C' \Sigma^{-1} \left[dY_t - \left(C \left(\overline{X}_t + T_2(\overline{X}_t - \hat{X}_t) \right) dt + \Sigma^{1/2} d\overline{V}_t \right) \right] \\
&= [A - P_t S] \overline{X}_t - [T_1(P_t) + P_t S T_2] (\overline{X}_t - \hat{X}_t) dt \\
&\quad + R^{1/2} d\overline{W}_t + P_t C' \Sigma^{-1} \left[dY_t - \Sigma^{1/2} d\overline{V}_t \right]
\end{aligned}$$

where $T_1 : \mathbb{S}_r^+ \mapsto \mathbb{M}_r$ stands for some mapping and T_2 some given matrix.

A key feature of this class of mean repulsion models is that their \mathcal{F}_t -conditional projections coincide with the Kalman-Bucy filter, only their conditional covariance matrices are altered.

To describe the Riccati equation associated with this class of nonlinear diffusions we observe that

$$d(\overline{X}_t - \hat{X}_t) = (A - P_t S - [T_1(P_t) + P_t S T_2]) (\overline{X}_t - \hat{X}_t) dt + R^{1/2} d\overline{W}_t - P_t C' R_2^{-1/2} d\overline{V}_t$$

Thus, the covariance evolution equation is given by the Riccati equation

$$\begin{aligned}
\partial_t P_t &= [A - P_t S (Id + T_2) - T_1(P_t)] P_t + P_t [A - P_t S (Id + T_2) - T_1(P_t)]' + R + P_t S P_t \\
&= A P_t + P_t A' + R - P_t S P_t - P_t S T_2 P_t - (T_1(P_t) P_t + P_t T_1(P_t)') - P_t T_2 S P_t
\end{aligned}$$

For instance, choosing

$$T_1(Q) = \epsilon_1 Q S \quad \text{and} \quad T_2 = \epsilon_2 Id$$

for some $\epsilon_i \geq 0$ we find that

$$\partial_t P_t = A P_t + P_t A' + R - P_t S_\epsilon P_t \quad \text{with} \quad S_{\epsilon_1, \epsilon_2} := (1 + 2(\epsilon_1 + \epsilon_2)) S$$

We let ϕ_t^ϵ be the Riccati semigroup associated with the above equation, with $\epsilon = (\epsilon_1, \epsilon_2) \in \Pi = [0, 1]^2$. Theorem 2.6 yields the following corollary.

Corollary 4.2. *There exists some $\delta \in [0, 1]$ such that for any $\epsilon = (\epsilon_1, \epsilon_2) \in [0, \delta]^2$ and for any time horizon $t \geq 0$ and any $Q \in \mathbb{S}_r^+$ we have*

$$\|\phi_t^\epsilon(Q) - \phi_t(Q)\|_2 \leq 2(\epsilon_1 + \epsilon_2) [\chi_1(\delta) + e^{-4t\nu} \chi_2(\delta, \|Q\|_2)]$$

for some finite constant $\chi_1(\delta)$, resp. $\chi_2(\delta, \|Q\|)$, whose values only depend on the parameter δ , resp. on $(\delta, \|Q\|)$. In addition, if $P_\epsilon = \phi_t^\epsilon(P_\epsilon)$ is the fixed point of ϕ_t^ϵ , then for any $\epsilon \in [0, \delta]$ we have

$$\|P_\epsilon - P\|_2 \leq 2(\epsilon_1 + \epsilon_2) \chi_1(\delta)$$

4.3 Block-diagonal localization

Assume that the covariance matrices associated with the Kalman filter given in (2) satisfy the following property

$$(H)_4 \quad \exists \iota > 0 \quad : \quad \forall t \geq 0 \quad |i - j| > \iota \implies P_t(i, j) = 0$$

In words, the coordinates of the signal have been arranged in such a way that the ι -long (or longer) range interactions between the state coordinates are null. Condition $(H)_4$ is met if and only if the matrices P_t are block-diagonal. Since the state variables are Gaussian, this property is equivalent to the fact that the state block components are two-by-two marginally independent. In this situation, the signal-observation process $(X_t, Y_t) = (X_t[k], Y_t[k])_{1 \leq k \leq n}$ defined in (1) can be decomposed into n -independent $(r[k] \times r'[k])$ -dimensional filtering problems $(X_t[k], Y_t[k])$ of the form

$$\begin{cases} dX_t[k] &= A[k] X_t[k] dt + R^{1/2}[k] dW_t[k] \\ dY_t[k] &= C[k] X_t[k] dt + \Sigma^{1/2}[k] dV_t[k] \end{cases} \quad \text{with } 1 \leq k \leq n.$$

with $r = \sum_{1 \leq i \leq n} r[i]$. In this elementary case, the resulting Kalman-Bucy filter and the associated Riccati equation collapse to n independent evolution equations. In this case, the drift and the sensor matrices (A, C) , as well as the covariance matrices (R, Σ) and P_t are block-diagonal matrices of appropriate dimensions.

Now observe that the sample covariance matrices $p_t(i, j)$ are generally *non-null* even if $P_t(i, j) = 0$. To mask these noisy entries, we use a localization mapping given in (40). It is readily checked that the mapping π satisfies the orthogonality condition $(H)_3$ discussed in (39) with the cellular algebra $\mathcal{B} = \mathcal{M}_{r[1]} \oplus \dots \oplus \mathcal{M}_{r[n]}$. With a little extra work, we can also check that

$$n^{-1} J \leq L \leq r^\star Id \implies n^{-1} Q \leq \pi(Q) \leq r^\star \text{Diag}(Q(1, 1), \dots, Q(r, r))$$

with $r^\star := \vee_{1 \leq k \leq n} r(k)$.

The central idea behind these mask-regularisations is to transform a given sample covariance matrix p into some covariance matrix with the same sparsity pattern as the limiting covariance P ; or in practice, to mask spurious ‘long-range’ correlations that are (almost) null in the true covariance. This idea is relevant in numerous applications of the EnKF in which state-space interaction and signal observations are mostly local, and a kind-of ‘decay-of-correlation’ effect is present; see [23, 20, 27].

One difficulty is ensuring the mask-matrix L is positive definite so that the projection $L \odot p$ is a positive map. In the block-diagonal model discussed above this property is clearly satisfied. In more general situations, several strategies can be underlined. The first one is to design mask-matrices as linear combinations $L = \sum_{i=1}^n l_i z_i z_i'$ of unit rank vectors z_i , with $l_i \geq 0$.

The sparsity structure in general settings is induced by the correlation structure of the signal-observation process; e.g. as illustrated in the simple block-diagonal case above. In practice, the sparsity structure (or ‘almost’ null entries) of the solution to the Riccati equation is generally difficult to extract from the signal and sensor models etc. In theory, any mask matrix L with $\{0, 1\}$ -entries such that $L \odot P_t = P_t$ can be used. Practically, a chosen mask matrix should reflect (as close as possible) the sparsity structure (or ‘almost’ null entries) of the covariance matrices to ensure effective performance and convergence.

4.4 Bose-Mesner projections

We introduce the Bose-Mesner algebra and relevant projections and applications here. For a more thorough discussion on Bose-Mesner algebras and their application in statistical and quantum

physics, combinatorics, coding, graph theory, and statistical covariance analysis (more particularly in experimental designs) we refer to the seminal article of Bose-Mesner [46], the ones of Nelder [51, 52], the more recent articles [53, 49, 48, 50], as well as the books [47, 44].

4.4.1 Association schemes

We set $\mathcal{I} = \{1, \dots, r\}$ the index set of the coordinates of the signal. Let $\mathcal{P} = \cup_{0 \leq q \leq n} \mathcal{P}_q$ be an n -partition of the product set \mathcal{I}^2 such that

- The associated classes \mathcal{P}_q are symmetric for any $0 \leq q \leq n$, and $\mathcal{P}_0 := \{(i, i) : i \in \mathcal{I}\}$.
- For any $0 \leq q_1, q_2 \leq n$, there exists some integer w_{q_1, q_2}^q (the parameters of the scheme; a.k.a. parameters of the first kind or the structural constants) such that

$$\forall 0 \leq q \leq n \quad \forall (i, j) \in \mathcal{P}_q \quad w_{q_1, q_2}^q = \text{Card} \{k \in \mathcal{I} : (i, k) \in \mathcal{P}_{q_1} \quad (k, j) \in \mathcal{P}_{q_2}\}$$

These association schemes can be interpreted as a partition of the edges/arcs of a complete graph (with vertex set \mathcal{I}) into n classes, often thought of as color classes. In this representation, there is a loop at each vertex and all the loops receive the same 0-th color. The number of triangles with a fixed arc-base with color q and the other two arcs with colors q_1 and q_2 is a number w_{q_1, q_2}^q that doesn't depend on the choice of the arc-base. Each vertex i is contained in exactly v_q arcs with color q . The number v_q is called the valency of the relation induced by \mathcal{P}_q . The parameters $w_{q_1, q_2}^q = w_{q_2, q_1}^q$ are called the parameters of the scheme (a.k.a. parameters of the first kind or the structural constants).

For each $1 \leq q \leq n$ we let B_q be the adjacency matrix; that is

$$B_q(k, l) = 1_{(k, l) \in \mathcal{P}_q} = B_q(l, k) \implies B_0 = Id \quad \text{and} \quad \sum_{0 \leq q \leq n} B_q = J$$

We also have

$$B_{q_1} B_{q_2} = B_{q_2} B_{q_1} = \sum_{0 \leq q \leq n} w_{q_1, q_2}^q B_q \quad \text{and} \quad B_q J = J B_q = v_q J$$

This shows that B_q has exactly v_q non-zero entries in every row and every column. Since for any $q_1 \neq q_2$ we have

$$(B_{q_1} \circ B_{q_2})(k, l) = 1_{(k, l) \in \mathcal{P}_{q_1} \cap \mathcal{P}_{q_2}} = 0 \implies B_{q_1} \circ B_{q_2} = 1_{q_1 = q_2} B_{q_1}$$

the set \mathcal{B} is also closed w.r.t. the Hadamard product and contains I, J . Thus, the set

$$\mathcal{B} := \left\{ \sum_{0 \leq q \leq n} b_q B_q : b = (b_q)_{0 \leq q \leq n} \subset \mathbb{R}^{n+1} \right\}$$

is an associative commutative algebra called the Bose-Mesner algebra of the association scheme. Notice that \mathcal{B} is also a matrix \star -algebra (i.e. closed by matrix multiplication, the transposition, addition and the scalar multiplication). These special cases of finite dimensional \mathbb{C}^\star -algebra are unitarily equivalent to block-diagonal matrices. By a theorem of Von Neumann we also mention that the orthogonal projection on any matrix \star -algebra is a positive map.

An illustration when $n = 2$ and $r = 6$ is provided by

$$B_0 = Id, \quad B_1 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \quad \text{and} \quad B_2 = J - [B_0 + B_1]$$

In this case we have $B_1^2 = 2Id + B_1 = 2B_0 + B_1$ and $B_1B_2 = 0 = B_2B_1$.

4.4.2 Minimal orthogonal projections

The commuting matrices B_q are simultaneously diagonalizable, \mathcal{B} has a basis of minimal orthogonal idempotents D_i ; that is, we have that

$$D_{q_1}D_{q_2} = 1_{q_1=q_2}D_{q_1} \quad \text{and} \quad \sum_{0 \leq q \leq r} D_q = Id$$

Without any loss of generality we can choose $D_0 = r^{-1}J$. The matrices D_q are called the minimal idempotents of the algebra \mathcal{B} . In addition, the column vectors $D_{i,1}, \dots, D_{i,r}$ of D_i are the eigenvectors of any matrix in \mathcal{B} . The eigenvector spaces $\mathcal{D}_i = \text{Span}(D_{i,1}, \dots, D_{i,r})$ are mutually orthogonal and every vector $u \in \mathbb{R}^r$ can be expressed uniquely as $u = \sum_{0 \leq q \leq n} u_i$ with $u_i \in \mathcal{D}_i$ (notice that \mathcal{D}_0 is the 1-dimensional space of constant vectors). Also notice that the dimension of \mathcal{D}_i equals to the rank of D_i , which is equal to the trace of D_i (since all non-zero eigenvalues of D_i are equal to 1).

In particular, we have

$$B_q D_k = \frac{\langle B_q, D_k \rangle_F}{\langle D_k, D_k \rangle_F} D_k \implies \lambda_k(B_q) = \frac{\langle B_q, D_k \rangle_F}{\langle D_k, D_k \rangle_F}$$

where $\lambda_k(B_q)$ stands for the k -th eigenvalue of B_q . Further details on these simultaneous diagonalization can be found in [45].

The orthogonal projection of a matrix Q on \mathcal{B} is given by the formulae

$$\pi(Q) = \text{proj}_{\mathcal{B}}(Q) := \sum_{0 \leq q \leq n} \frac{\langle Q, B_q \rangle_F}{\langle B_q, B_q \rangle_F} B_q = \sum_{0 \leq q \leq n} \frac{\langle Q, D_q \rangle_F}{\langle D_q, D_q \rangle_F} D_q$$

To check condition (H)₃ we observe that

$$D_{q_1}D_{q_2} = 1_{q_1=q_2}D_{q_1} \implies \text{proj}_{\mathcal{B}}(D_q Q) = \frac{\langle Q, D_q \rangle_F}{\langle D_q, D_q \rangle_F} D_q = D_q \text{proj}_{\mathcal{B}}(Q)$$

This yields

$$\forall B \in \mathcal{B} \quad \text{proj}_{\mathcal{B}}(B [Q - \text{proj}_{\mathcal{B}}(Q)]) = 0$$

For any matrix M we have

$$\langle MM', D_q \rangle_F = \text{tr}(D_q MM') = \text{tr}(M' D_q^2 M) = \text{tr}((D_q M)'(D_q M)) \geq 0$$

This implies that

$$\forall Q \in \mathbb{S}_r^+ \quad \langle Q, D_q \rangle_F \geq 0 \quad \text{and} \quad \text{proj}_{\mathcal{B}}(Q) = \sum_{0 \leq q \leq n} \frac{\langle Q, D_q \rangle_F}{\langle D_q, D_q \rangle_F} D_q \geq 0$$

This shows that the orthogonal projection is a positive map from the algebra of square matrices into itself. In addition, it is trace-preserving and unital in the sense that

$$\mathrm{tr}(\mathrm{proj}_{\mathcal{B}}(Q)) = \mathrm{tr}(Q) \quad \text{and} \quad \mathrm{proj}_{\mathcal{B}}(Id) = Id$$

Last, but not least, using the decomposition

$$Q = \mathrm{proj}_{\mathcal{B}}(Q) + [Q - \mathrm{proj}_{\mathcal{B}}(Q)] \implies \|Q - \mathrm{proj}_{\mathcal{B}}(Q)\|_F \leq \|Q\|_F \leq \mathrm{tr}(Q) \quad (49)$$

as soon as $Q \in \mathbb{S}_r^+$. Working a little harder, we check that

$$\|Q - \mathrm{proj}_{\mathcal{B}}(Q)\|_F \leq \|Q\|_F \left[1 - \frac{1}{n+1} \frac{1}{\wedge_{0 \leq q \leq n} \mathrm{tr}(D_q)} \frac{\mathrm{tr}(Q)^2}{\mathrm{tr}(Q^2)} \right]^{1/2}$$

4.4.3 Distance regular graphs

Another prototype of Bose-Mesner algebra are distance regular graphs. Given a connected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with vertex set \mathcal{V} and arc/edges set \mathcal{E} , we let $\rho(i, j)$ be the path-length distance between two vertices $i, j \in \mathcal{V}$. Let

$$\mathcal{S}(i, q) = \{j \in \mathcal{V} : \rho(i, j) = q\}$$

be the sphere of radius q . The graph \mathcal{G} is distance regular if and only if we have

$$\mathrm{Card}(\mathcal{S}(i, q_1) \cap \mathcal{S}(j, q_2)) = w_{q_1, q_2}^{\rho(i, j)}$$

for some parameters w_{q_1, q_2}^q . In words, for every two vertices (i, j) at distance q there are precisely w_{q_1, q_2}^q vertices in the graph at distance q_1 from i and q_2 from j .

In these settings, the matrices

$$(B_q)(i, j) = 1_{\rho(i, j) = q} \quad \text{with} \quad 0 \leq q \leq \mathrm{diam}(\mathcal{G}) := \sup_{(i, j) \in \mathcal{V}^2} \rho(i, j)$$

are called the distance matrices ($B_0 = Id$, B_1 the adjacency matrix, and so on). In this situation, the association scheme is given by the partition

$$\forall 0 \leq k \leq d := \mathrm{diam}(\mathcal{G}) \quad \mathcal{P}_k = \{(i, j) \in \mathcal{I}^2 : \rho(i, j) = k\}$$

In addition we have $w_{q_2, 1}^{q_1} = 0$ for any $q_1 \neq 0$, and $q_2 \neq \{q_1 - 1, q_1, q_1 + 1\}$. If we set

$$a_q := w_{q, q}^0 \quad b_q := w_{q-1, 1}^q \quad \text{and} \quad c_q := w_{q+1, 1}^q$$

then we have

$$B_1 B_q = c_{q-1} B_{q-1} + (a_1 - b_q - c_q) B_q + b_{q+1} B_{q+1}$$

and

$$B_1 B_d = c_{d-1} B_{d-1} + (a_1 - b_d) B_d$$

This shows that the adjacency matrix B_1 generates \mathcal{B} (i.e. the matrices B_q can be written as polynomials of degree q in B_1), so that the eigenvalues $(\lambda_k(B_1))_{1 \leq k \leq d}$ of B_1 are mutually distinct.

4.4.4 Riccati solvers

A given matrix Q belongs to \mathcal{B} if and only if it is constant within each block. To check this claim, we observe that

$$\begin{aligned} Q = \text{proj}_{\mathcal{B}}(Q) &\iff \forall 0 \leq q \leq n \quad Q \odot B_q = \frac{\langle Q, B_q \rangle_F}{\langle B_q, B_q \rangle_F} B_q \\ &\iff \forall 0 \leq q \leq n \quad \forall (i, j) \in \mathcal{P}_q \quad \sum_{(k, l) \in \mathcal{P}_q} Q_{k, l} = w_{q, q}^0 Q_{i, j} \\ &\iff \forall 0 \leq q \leq n \quad \forall (i, j), (i', j') \in \mathcal{P}_q \quad Q_{i, j} = Q_{i', j'} \end{aligned}$$

In other words, the matrix is constant within each block. When $r = r'$, then $(\pi(A), \pi(S), \pi(R)) = (A, R, S)$ is satisfied as soon as $(A, R, C, \Sigma^{-1}) \in \mathcal{B}$.

We further assume that $(A, R, S) \in \mathcal{B}$ and we set

$$A := \sum_{0 \leq q \leq n} a_q D_q \quad R := \sum_{0 \leq q \leq n} r_q D_q \quad \text{and} \quad S := \sum_{0 \leq q \leq n} s_q D_q$$

Let $P_0 = \pi(P_0) = \sum_{0 \leq q \leq n} \alpha_q(0) D_q$ be some covariance matrix in \mathcal{B} . By Theorem 2.12 we have

$$P_t = \pi(P_t) = \sum_{0 \leq q \leq n} \alpha_q(t) D_q$$

In addition, we have

$$\begin{aligned} \partial_t P_t &= \sum_{0 \leq q \leq n} \partial_t \alpha_q(t) D_q = \text{Ricc}(P_t) = A\pi(P_t) - \pi(P_t)A' + R - \pi(P_t)S\pi(P_t) \\ &= \sum_{0 \leq q \leq n} [2a_q \alpha_q(t) + r_q - \alpha_q(t)^2 s_q] D_q \end{aligned}$$

This implies that

$$\begin{cases} \partial_t \alpha_q(t) &= 2a_q \alpha_q(t) + r_q - \alpha_q(t)^2 s_q \\ q &= 0, \dots, n \end{cases}$$

When $s_q \neq 0 \neq r_q$ this collection of Riccati equations take the form

$$\partial_t \alpha_q(t) = -s_q (\alpha_q(t) - z_1(q)) (\alpha_q(t) - z_2(q))$$

with the couple of roots

$$z_1(q) = \frac{a_q - \sqrt{a_q^2 + s_q r_q}}{s_q} < 0 < z_2(q) = \frac{a_q + \sqrt{a_q^2 + s_q r_q}}{s_q}.$$

The solutions of the above equations are given by the formulae:

$$\alpha_q(t) - z_2 = (\alpha_q(0) - z_2(q)) \frac{(z_2(q) - z_1(q)) e^{-2t\sqrt{a_q^2 + s_q r_q}}}{(z_2(q) - \alpha_q(0)) e^{-2t\sqrt{a_q^2 + s_q r_q}} + (\alpha_q(0) - z_1(q))} \xrightarrow{t \rightarrow \infty} 0$$

4.5 Stein-Shrinkage models

Stein-Shrinkage models are an extension of the variation inflation model to parameters $\epsilon = \epsilon(Q)$ and target-type matrices $T = T(Q)$ that both may depend on the matrix Q . These models are defined by the formula

$$\pi(Q) = \epsilon(Q) T(Q) + (1 - \epsilon(Q)) Q$$

for some function $Q \mapsto \epsilon(Q) \in [0, 1]$ and some mapping T from \mathbb{S}_r^+ into itself. It is not within scope of this article to review all the relevant covariance matrix estimators encountered in the statistics literature fitting this general model. We will just illustrate this model with three important and currently used approximations:

- *Mask matrix estimates* are associated with mappings T defined by $T(Q) := L \odot Q$ with a matrix L of the form

$$L_{i,j} = 1_{|i-j| < \iota} \implies Q - L \odot Q = 1_{|i-j| \geq \iota} Q_{i,j} \quad (50)$$

- *Maximum likelihood* type estimates are associated with mappings T defined by

$$T(Q) := \operatorname{argmax}_{q \in \mathbb{S}_r^+} (\log \det(q) + \operatorname{tr}(q^{-1}Q) + \alpha \|L \odot q\|)$$

for some $\alpha > 0$, some mask matrix L [56, 57, 62, 63], and some matrix norm $\|\cdot\|$ on \mathbb{S}_r^+ .

- *Nyström estimates* are associated with mappings T defined by

$$T(Q) = (J - L_{\mathcal{P}^c}) \odot Q + L_{\mathcal{P}^c} \odot [Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}] \quad (51)$$

where $\{1, \dots, r\} = \mathcal{P} \cup \mathcal{P}^c$ stands for a partition of the index coordinate set and $L_{\mathcal{P}^c}$ stands for the mask matrix defined by

$$L_{\mathcal{P}^c}(i, j) = 1_{\mathcal{P}^c \times \mathcal{P}^c}(i, j)$$

At the level of the sample covariance matrices p_0 , the matrix $T(p_0)$ is obtained by taking the sample covariance matrix associated with projection $\mathcal{T}_{\mathcal{V}}(\zeta_l)$ of the state particle vectors

$$\zeta' := \begin{bmatrix} \zeta'_1 \\ \vdots \\ \zeta'_r \end{bmatrix} := [\xi_0^1 - m_0, \dots, \xi_0^N - m_0] = \begin{bmatrix} \xi_0^1(1) - m_0(1) & \dots & \xi_0^N(1) - m_0(1) \\ \vdots & \vdots & \vdots \\ \xi_0^1(r) - m_0(r) & \dots & \xi_0^N(r) - m_0(r) \end{bmatrix}$$

onto the vector space $\mathcal{V}_{\mathcal{P}}$ of \mathbb{R}^N spanned by the random vectors

$$V_i = \zeta_{k_i} := \begin{bmatrix} \xi_0^1(k_i) - m_0(k_i) \\ \vdots \\ \xi_0^N(k_i) - m_0(k_i) \end{bmatrix} \in \mathbb{R}^N \quad \text{with } \mathcal{P} = \{k_1, \dots, k_s\} \text{ and } s := \operatorname{Card}(\mathcal{P}) \leq r$$

More precisely, if we set

$$\begin{aligned} N T(p_0) &= \begin{bmatrix} (\mathcal{T}_{\mathcal{V}}\zeta_l)' \\ \vdots \\ (\mathcal{T}_{\mathcal{V}}\zeta_r)' \end{bmatrix} [\mathcal{T}_{\mathcal{V}}\zeta_l, \dots, \mathcal{T}_{\mathcal{V}}\zeta_r] \\ &= (\mathcal{T}_{\mathcal{V}}\zeta)' \mathcal{T}_{\mathcal{V}}\zeta = \zeta' \mathcal{T}_{\mathcal{V}}\zeta = \begin{bmatrix} \langle \mathcal{T}_{\mathcal{V}}\zeta_1, \mathcal{T}_{\mathcal{V}}\zeta_1 \rangle & \dots & \langle \mathcal{T}_{\mathcal{V}}\zeta_1, \mathcal{T}_{\mathcal{V}}\zeta_r \rangle \\ \vdots & & \vdots \\ \langle \mathcal{T}_{\mathcal{V}}\zeta_r, \mathcal{T}_{\mathcal{V}}\zeta_1 \rangle & \dots & \langle \mathcal{T}_{\mathcal{V}}\zeta_r, \mathcal{T}_{\mathcal{V}}\zeta_r \rangle \end{bmatrix} \end{aligned}$$

then we have that

$$\mathbb{E}(T(p_0)) = T(P_0) + \frac{s}{N} L_{\mathcal{P}^c} \odot [Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}] \quad (52)$$

The proof of this bias property and related variance estimates can be found in [54]. For the convenience of the reader a proof of the last assertion is provided in the appendix on page 44.

For mask type mappings of the form (50), condition $(H)_0$ is satisfied by first letting

$$(B_0, B_1, B_2) = (0, 0, 0) \implies \Gamma_\pi(Q) = \mathcal{R}(Q) := \epsilon(Q)^2 (L \odot Q - Q) S (L \odot Q - Q)$$

To ensure the uniform estimate $\sup_{Q \in \mathbb{S}_r^+} \|\mathcal{R}(Q)\|_2 < \infty$ holds we use Gershgorin's theorem to show that

$$\|Q - L \odot Q\|_2 \leq l_\iota(Q) := \sup_{1 \leq i \leq r} \sum_{|i-j| \geq \iota} |Q_{i,j}|$$

This yields

$$\|\mathcal{R}(Q)\|_2 \leq \epsilon^2(Q) \|S\|_2 l_\iota^2(Q) \implies \mathcal{R}(Q) \leq \overline{\mathcal{R}}(Q) Id \quad \text{with} \quad \overline{\mathcal{R}}(Q) = \epsilon^2(Q) \|S\|_2 l_\iota^2(Q)$$

When $l_\iota(Q)$ is too large, the quadratic perturbation may have some destabilizing effects. To avoid these issues we assume that $\epsilon(Q)$ is chosen so that

$$\epsilon(Q) = \epsilon_1 1_{l_\iota(Q) \leq \epsilon_2^{-1}} \implies \mathcal{R}(Q) \leq \varpi Id \quad \text{with} \quad \varpi = \|S\|_2 (\epsilon_1/\epsilon_2)^2$$

for some $\epsilon_1 \in [0, 1]$, and some threshold $\epsilon_2 > 0$. In this case, condition $(H)_1$ is also met with

$$R_\pi = R + (\epsilon_1/\epsilon_2)^2 \|S\|_2 Id \quad A_\pi = A \quad \text{and} \quad S_\pi = S \implies \Xi_\pi(Q) = (\epsilon_1/\epsilon_2)^2 \|S\|_2 Id$$

Arguing as in the end of Section 4.1 we have

$$\begin{aligned} R^{-1/2} R_\pi R^{-1/2} - Id &= (\epsilon_1/\epsilon_2)^2 \|S\|_2 R^{-1} \\ \implies R^{-1/2} R_\pi R^{-1/2} &\leq (1 + (\epsilon_1/\epsilon_2)^2 \|S\|_2 \|R^{-1}\|) Id \\ \implies R &\leq R_\pi \leq R (1 + (\epsilon_1/\epsilon_2)^2 \|S\|_2 \|R^{-1}\|) Id \implies (H)_2 \end{aligned}$$

Now we can consider the set

$$\Pi = \{\pi_{\epsilon_1, \epsilon_2} : (\epsilon_1, \epsilon_2) \in ([0, 1] \times [\delta, \delta^{-1}])\}$$

for some given parameter δ and the just described mappings $\pi_{\epsilon_1, \epsilon_2}$ given by

$$\pi_{\epsilon_1, \epsilon_2}(Q) = Q + \epsilon_1 1_{l_\iota(Q) \leq \epsilon_2^{-1}} [L \odot Q - Q] \implies \|\pi_{\epsilon_1, \epsilon_2} - id\|_2 \leq \epsilon_1/\epsilon_2$$

The associated δ -balls around the identity mapping are given in this case by the compact sets

$$B(\delta) = \{\pi_{\epsilon_1, \epsilon_2} : \epsilon_1/\epsilon_2 \leq \delta\}$$

for any $\delta \leq 1$.

More generally, the Stein-Shrinkage models discussed above can be extended without further work to general mappings of the following form

$$\pi_{\epsilon_1, \epsilon_2}(Q) = Q + \epsilon_1 1_{l_T(Q) \leq \epsilon_2^{-1}} [T(Q) - Q]$$

where T stands for some mapping from \mathbb{S}_r^+ into itself such that

$$\|T(Q) - Q\|_2 \leq l_T(Q)$$

for some mapping $Q \in \mathbb{S}_r^+ \mapsto l_T(Q) \in [0, \infty[$. Further examples of such mappings include the Bose-Mesner projections $T(Q) = \text{proj}_B(Q)$ discussed in Section 4.4 and which can be seen to fit this model via the trace operator in (49).

In this general setting, Theorem 2.6 yields the following corollary.

Corollary 4.3. *There exists some $\rho \in [0, 1]$ such that for any $(\epsilon_1, \epsilon_2) \in ([0, 1] \times [\delta, \delta^{-1}])$ with $\epsilon_1 \leq \rho \epsilon_2$, for any $Q \in \mathbb{S}_r^+$ and any time horizon $t \geq 0$ we have*

$$\|\phi_t^{\pi_{\epsilon_1, \epsilon_2}}(Q) - \phi_t(Q)\|_2 \leq (\epsilon_1/\epsilon_2)^2 [\chi_1(\rho) + e^{-4t\nu} \chi_2(\rho, \|Q\|_2)]$$

for some finite constant $\chi_1(\rho)$, resp. $\chi_2(\rho, \|Q\|)$, whose values only depend on the parameter δ , resp. on $(\rho, \|Q\|)$. In addition, for any $\epsilon_1 \leq \rho \epsilon_2$ we have

$$\|P_{\pi_{\epsilon_1, \epsilon_2}} - P\|_2 \leq (\epsilon_1/\epsilon_2)^2 \chi_1(\rho)$$

This section illustrates how our first class of perturbation-type model captures most projection-type mappings; and consequently those results relevant to perturbation-type mappings are applicable to projection-type models (but not vice-versa).

Appendix

Proof of formula (8)

Let $\phi_{s,t}^\pi$ be the semigroup of equation (8). Also let \overline{X}_t^π be the time non-homogeneous diffusion given by the equation

$$\begin{aligned} d\overline{X}_t^\pi &= A \overline{X}_t^\pi dt + R^{1/2} d\overline{W}_t + \pi(\phi_{0,t}^\pi(Q)) C' \Sigma^{-1} \left[dY_t - (C \overline{X}_t^\pi dt + \Sigma^{1/2} d\overline{V}_t) \right] \\ &= [A - \pi(\phi_{0,t}^\pi(Q)) S] \overline{X}_t^\pi dt + \pi(\phi_{0,t}^\pi(Q)) C' \Sigma^{-1} dY_t + dM_t^\pi \end{aligned}$$

with the r -valued martingale

$$dM_t^\pi := R^{1/2} d\overline{W}_t - \pi(\phi_{0,t}^\pi(Q)) C' R_2^{-1/2} d\overline{V}_t$$

with covariation matrix

$$\partial_t \langle M^\pi(k), M^\pi(l) \rangle_t = [R + \pi(\phi_{0,t}^\pi(Q)) S \pi(\phi_{0,t}^\pi(Q))] (k, l)$$

We have

$$\begin{aligned} \overline{X}_t^\pi &= \exp \left(\int_0^t [A - \pi(\phi_{0,s}^\pi(Q)) S] ds \right) \overline{X}_0^\pi \\ &\quad + \int_0^t \exp \left(\int_s^t [A - \pi(\phi_{0,u}^\pi(Q)) S] du \right) \pi(\phi_{0,s}^\pi(Q)) C' \Sigma^{-1} dY_s \\ &\quad + \int_0^t \exp \left(\int_s^t [A - \pi(\phi_{0,u}^\pi(Q)) S] du \right) dM_s^\pi \end{aligned}$$

This implies that the conditional expectations $\hat{X}_t^\pi = \mathbb{E}(\bar{X}_t^\pi \mid \mathcal{F}_t)$ are given by the formula

$$\begin{aligned} \hat{X}_t^\pi &= \exp \left(\int_0^t [A - \pi(\phi_{0,s}^\pi(Q)) S] ds \right) \hat{X}_0^\pi \\ &\quad + \int_0^t \exp \left(\int_s^t [A - \pi(\phi_{0,u}^\pi(Q)) S] du \right) \pi(\phi_{0,s}^\pi(Q)) C' \Sigma^{-1} dY_s \end{aligned}$$

Equivalently, we have

$$d\hat{X}_t^\pi = A \hat{X}_t^\pi dt + \pi(\phi_{0,t}^\pi(Q)) C' \Sigma^{-1} [dY_t - C \hat{X}_t^\pi dt]$$

from which we prove that

$$d[\bar{X}_t^\pi - \hat{X}_t^\pi] = [A - \pi(\phi_{0,t}^\pi(Q)) S] [\bar{X}_t^\pi - \hat{X}_t^\pi] dt + dM_t^\pi$$

This implies that the covariation matrices

$$Q_t^\pi := \mathbb{E} \left([\bar{X}_t^\pi - \hat{X}_t^\pi] [\bar{X}_t^\pi - \hat{X}_t^\pi]' \mid \mathcal{F}_t \right) = \mathbb{E} \left([\bar{X}_t^\pi - \hat{X}_t^\pi] [\bar{X}_t^\pi - \hat{X}_t^\pi]' \right)$$

don't depend on the observation process, and they satisfy the equation

$$\partial_t Q_t^\pi = [A - \pi(\phi_{0,t}^\pi(Q)) S] Q_t^\pi + Q_t^\pi [A - \pi(\phi_{0,t}^\pi(Q)) S]' + R + \pi(\phi_{0,t}^\pi(Q)) S \pi(\phi_{0,t}^\pi(Q))$$

Recalling that $\phi_{0,t}^\pi(Q)$ is the Riccati semigroup of the equation (8) we have

$$\partial_t (Q_t^\pi - \phi_{0,t}^\pi(Q)) = [A - \pi(\phi_{0,t}^\pi(Q)) S] (Q_t^\pi - \phi_{0,t}^\pi(Q)) + (Q_t^\pi - \phi_{0,t}^\pi(Q)) [A - \pi(\phi_{0,t}^\pi(Q)) S]'$$

We conclude that

$$Q_0^\pi = Q \implies Q_t^\pi = \phi_{0,t}^\pi(Q) = \mathcal{P}_{\eta_t^\pi} \implies \pi(Q_t^\pi) = \pi(\phi_{0,t}^\pi(Q))$$

where $\eta_t^\pi = \text{Law}(\bar{X}_t^\pi \mid \mathcal{F}_t)$. This ends the proof of (8). See also [11, page 242] (among numerous other sources) for the related covariance flow of a Kalman filter with an arbitrary gain matrix.

Proof of Lemma 2.10

Condition (37) implies that

$$\begin{aligned} \lambda_{\min}(\mathcal{W}_t(\pi_2)) \geq \varpi_{-,t}(\pi) &\implies \lambda_{\min}(\mathcal{W}_t(\pi_2)^{1/2}) \geq \sqrt{\varpi_{-,t}(\pi_2)} \\ &\implies \lambda_{\max}(\mathcal{W}_t(\pi_2)^{-1/2}) \leq 1/\sqrt{\varpi_{-,t}(\pi_2)} \end{aligned}$$

from which we conclude that

$$\|\mathcal{W}_t(\pi_2)^{-1/2}\|_2^2 \leq \varpi_{-,t}(\pi_2)^{-1} \quad (53)$$

We also have

$$(37) \implies \mathcal{W}_t(\pi_1) \geq \mathcal{W}_t(\pi_1, \pi_2) := \int_0^t \mathcal{U}_s(\pi_1) \mathcal{V}_s(\pi_2) \mathcal{U}_s'(\pi_1) ds \quad (54)$$

Observe that

$$\partial_s \mathcal{W}_s(\pi_1, \pi_2) = \mathcal{U}_s(\pi_1, \pi_2) [\partial_s \mathcal{W}_s(\pi_2)] \mathcal{U}_s(\pi_1, \pi_2)'$$

with the flow of matrices

$$\mathcal{U}_s(\pi_1, \pi_2) = \mathcal{U}_s(\pi_1) \mathcal{U}_s(\pi_2)^{-1} \implies \mathcal{U}_s(\pi, \pi) = Id$$

We set

$$\|\mathcal{U}\|_2 := \sup_{(s, \pi) \in ([0, t] \times \Pi)} \|\mathcal{U}_s(\pi)\|_2 < \infty \quad \text{and} \quad \|\mathcal{V}\|_2 := \sup_{(s, \pi) \in ([0, t] \times \Pi)} \|\mathcal{V}_s(\pi)\|_2 < \infty$$

In this notation, using the fact that

$$\sup_{s \in [0, t]} \|\partial_s \mathcal{W}_s(\pi_2)\|_2 \leq t \|\mathcal{U}\|_2^2 \|\mathcal{V}\|_2$$

we find that

$$\sup_{s \in [0, t]} \|\partial_s \mathcal{W}_s(\pi_1, \pi_2) - \partial_s \mathcal{W}_s(\pi_2)\|_2 \leq c_{\mathcal{U}} \|\pi_1 - \pi_2\| t \|\mathcal{U}\|_2 \|\mathcal{V}\|_2 [2 + c_{\mathcal{U}} \|\pi_1 - \pi_2\| t \|\mathcal{U}\|_2 \|\mathcal{V}\|_2]$$

from which we conclude that

$$\|\mathcal{W}_s(\pi_1, \pi_2) - \mathcal{W}_s(\pi_2)\|_2 \leq c_{\mathcal{U}} \|\pi_1 - \pi_2\| t^2 \|\mathcal{U}\|_2 \|\mathcal{V}\|_2 [2 + c_{\mathcal{U}} \|\pi_1 - \pi_2\| t^2 \|\mathcal{U}\|_2 \|\mathcal{V}\|_2] \quad (55)$$

The inequality in (54) implies that

$$\begin{aligned} \mathcal{W}_t(\pi_1)^{-1} &\leq \mathcal{W}_t(\pi_2)^{-1/2} \left[\mathcal{W}_t(\pi_2)^{1/2} \mathcal{W}_t(\pi_1, \pi_2)^{-1} \mathcal{W}_t(\pi_2)^{1/2} \right] \mathcal{W}_t(\pi_2)^{-1/2} \\ &= \mathcal{W}_t(\pi_2)^{-1/2} \left[\mathcal{W}_t(\pi_2)^{-1/2} \mathcal{W}_t(\pi_1, \pi_2) \mathcal{W}_t(\pi_2)^{-1/2} \right]^{-1} \mathcal{W}_t(\pi_2)^{-1/2} \end{aligned}$$

On the other hand we have

$$\begin{aligned} &\left[\mathcal{W}_t(\pi_2)^{-1/2} \mathcal{W}_t(\pi_1, \pi_2) \mathcal{W}_t(\pi_2)^{-1/2} \right]^{-1} \\ &= \left[Id - \mathcal{W}_t(\pi_2)^{-1/2} \{ \mathcal{W}_t(\pi_2) - \mathcal{W}_t(\pi_1, \pi_2) \} \mathcal{W}_t(\pi_2)^{-1/2} \right]^{-1} \end{aligned}$$

This yields the estimate

$$\mathcal{W}_t(\pi_2)^{1/2} \mathcal{W}_t(\pi_1)^{-1} \mathcal{W}_t(\pi_2)^{1/2} \leq \sum_{n \geq 0} \left[\mathcal{W}_t(\pi_2)^{-1/2} \{ \mathcal{W}_t(\pi_2) - \mathcal{W}_t(\pi_1, \pi_2) \} \mathcal{W}_t(\pi_2)^{-1/2} \right]^n$$

Combining (53) with (55), for any $\epsilon > 0$ there exists some $\delta(t, \epsilon, \pi_2) > 0$ such that

$$\|\pi_1 - \pi_2\| \leq \delta(t, \epsilon) \implies \|\mathcal{W}_t(\pi_2)^{-1/2} \{ \mathcal{W}_t(\pi_2) - \mathcal{W}_t(\pi_1, \pi_2) \} \mathcal{W}_t(\pi_2)^{-1/2}\|_2 \leq 1 - \epsilon$$

This ends the proof of the lemma. ■

Proof of the bias estimate (52)

Observe that if $Z \sim \mathcal{N}(0, Q)$ is Gaussian, then the conditional distribution of $Z_{\mathcal{P}^c} = (Z_k)_{k \in \mathcal{P}^c}$ given $Z_{\mathcal{P}} = (Z_k)_{k \in \mathcal{P}}$ is again a centred Gaussian with covariance matrix

$$T_{\mathcal{P}}(Q) = Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}$$

where $Q_{\mathcal{P}}^{-1}$ stands for the Moore-Penrose pseudo-inverse of $Q_{\mathcal{P}}$. The matrix $T_{\mathcal{P}}(Q)$ can be seen as the Schur complement of $Q_{\mathcal{P}}$ in Q . This shows that

$$Q - T(Q) = L_{\mathcal{P}^c} \odot T_{\mathcal{P}}(Q)$$

In this notation we have

$$\begin{aligned} \zeta \zeta' &= [\xi_0^1 - m_0, \dots, \xi_0^N - m_0] \begin{bmatrix} (\xi_0^1 - m_0)' \\ \vdots \\ (\xi_0^N - m_0)' \end{bmatrix} = \sum_{1 \leq i \leq N} (\xi_0^i - m_0)(\xi_0^i - m_0)' \\ &= \begin{bmatrix} \zeta_1 \\ \vdots \\ \zeta_r \end{bmatrix} [\zeta_1', \dots, \zeta_r'] = \begin{bmatrix} \langle \zeta_1, \zeta_1 \rangle & \dots & \langle \zeta_1, \zeta_r \rangle \\ \vdots & & \vdots \\ \langle \zeta_r, \zeta_1 \rangle & \dots & \langle \zeta_r, \zeta_r \rangle \end{bmatrix} \end{aligned}$$

We let g be the matrix

$$\forall 1 \leq i, j \leq s \quad g_{i,j} := \langle V_i, V_j \rangle \iff g = \begin{bmatrix} V_1' \\ \vdots \\ V_s' \end{bmatrix} [V_1, \dots, V_s]$$

Also let $g^- = (g^{i,j})_{1 \leq i, j \leq s}$ be the pseudo-inverse of g . The orthogonal projection of a vector ζ_l with $l \notin \mathcal{P}$ is given by

$$\text{proj}_{\mathcal{V}}(\zeta_l) = \sum_{1 \leq i \leq s} \langle \sum_{1 \leq j \leq s} g^{i,j} V_j, \zeta_l \rangle V_i = [V_1, \dots, V_s] g^- \begin{bmatrix} V_1' \\ \vdots \\ V_s' \end{bmatrix} \zeta_l := \mathcal{T}_{\mathcal{V}} \zeta_l$$

$$\begin{aligned} N T(p_0) &= \begin{bmatrix} \mathcal{T}_{\mathcal{V}} \zeta_l \\ \vdots \\ \mathcal{T}_{\mathcal{V}} \zeta_r \end{bmatrix} [(\mathcal{T}_{\mathcal{V}} \zeta_l)', \dots, (\mathcal{T}_{\mathcal{V}} \zeta_r)'] = \begin{bmatrix} \langle \mathcal{T} \zeta_1, \mathcal{T}_{\mathcal{V}} \zeta_1 \rangle & \dots & \langle \mathcal{T} \zeta_1, \mathcal{T}_{\mathcal{V}} \zeta_r \rangle \\ \vdots & & \vdots \\ \langle \mathcal{T} \zeta_r, \mathcal{T}_{\mathcal{V}} \zeta_1 \rangle & \dots & \langle \mathcal{T} \zeta_r, \mathcal{T}_{\mathcal{V}} \zeta_r \rangle \end{bmatrix} \\ &= \begin{bmatrix} \langle \zeta_1, \mathcal{T}_{\mathcal{V}} \zeta_1 \rangle & \dots & \langle \zeta_1, \mathcal{T}_{\mathcal{V}} \zeta_r \rangle \\ \vdots & & \vdots \\ \langle \zeta_r, \mathcal{T}_{\mathcal{V}} \zeta_1 \rangle & \dots & \langle \zeta_r, \mathcal{T}_{\mathcal{V}} \zeta_r \rangle \end{bmatrix} = \zeta (\mathcal{T}_{\mathcal{V}} \zeta)' \end{aligned}$$

Given \mathcal{V} , the N random vectors $\zeta_{\mathcal{P}^c}^i = (\zeta_k^i)_{k \notin \mathcal{P}} \in \mathbb{R}^{r-s}$, with $1 \leq i \leq N$ are independent random vectors in \mathbb{R}^N with mean

$$\mathbb{E}(\zeta_{\mathcal{P}^c}^i | \mathcal{V}) = Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} \zeta_{\mathcal{P}}^i \quad \text{with} \quad \zeta_{\mathcal{P}}^i := (\zeta_k^i)_{k \in \mathcal{P}} \in \mathbb{R}^s$$

and covariance matrix

$$\mathbb{E} \left([\zeta_{\mathcal{P}^c}^i - \mathbb{E}(\zeta_{\mathcal{P}^c}^i | \mathcal{V})] [\zeta_{\mathcal{P}^c}^i - \mathbb{E}(\zeta_{\mathcal{P}^c}^i | \mathcal{V})]' | \mathcal{V} \right) = Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}$$

This implies that for any $k, l \notin \mathcal{P}$ we have

$$\begin{aligned}
\mathbb{E}(\langle \zeta_k, \mathcal{T}_V \zeta_l \rangle | \mathcal{V}) &= \sum_{1 \leq i, j \leq N} \mathbb{E} \left(\zeta_k^i \mathcal{T}_V(i, j) \zeta_l^j | \mathcal{V} \right) \\
&= \sum_{1 \leq i, j \leq N} \mathbb{E} \left([\zeta_k^i - \mathbb{E}(\zeta_k^i | \mathcal{V})] \mathcal{T}_V(i, j) [\zeta_l^j - \mathbb{E}(\zeta_l^j | \mathcal{V})] | \mathcal{V} \right) \\
&\quad + \sum_{1 \leq i, j \leq N} \mathbb{E}(\zeta_k^i | \mathcal{V}) \mathcal{T}_V(i, j) \mathbb{E}(\zeta_l^j | \mathcal{V}) \\
&= \text{tr}(\mathcal{T}_V) (Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c})(k, l) \\
&\quad + \sum_{1 \leq i, j \leq N} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} \zeta_{\mathcal{P}}^i)(k) \mathcal{T}_V(i, j) (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} \zeta_{\mathcal{P}}^j)(l)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{1 \leq i, j \leq N} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} \zeta_{\mathcal{P}}^i)(k) \mathcal{T}_V(i, j) (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} \zeta_{\mathcal{P}}^j)(l) \\
&= \sum_{u, v \in \mathcal{P}} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(k, u) \sum_{1 \leq i, j \leq N} \begin{bmatrix} \zeta_u^i & \mathcal{T}_V(i, j) & \zeta_v^j \end{bmatrix} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(l, v) \\
&= \sum_{u, v \in \mathcal{P}} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(k, u) \langle \zeta_u, \mathcal{T}_V \zeta_v \rangle (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(l, v) \\
&= \sum_{u, v \in \mathcal{P}} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(k, u) \langle \zeta_u, \zeta_v \rangle (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(l, v)
\end{aligned}$$

Taking the expectation we find that

$$\begin{aligned}
\mathbb{E}(\langle \zeta_k, \mathcal{T}_V \zeta_l \rangle) &= s (Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c})(k, l) \\
&\quad + N \sum_{u, v \in \mathcal{P}} (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(k, u) Q_{\mathcal{P}}(u, v) (Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1})(l, v) \\
&= s (Q_{\mathcal{P}^c} - Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c})(k, l) + N [Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}](k, l) \\
&= \{s Q_{\mathcal{P}^c} + (N - s) [Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}]\}(k, l)
\end{aligned}$$

This shows that

$$\begin{aligned}
(51) \implies \mathbb{E}(T(p_0)) &= (J - L_{\mathcal{P}^c}) \odot Q \\
&\quad + L_{\mathcal{P}^c} \odot \left(\frac{s}{N} Q_{\mathcal{P}^c} + \left(1 - \frac{s}{N}\right) [Q_{\mathcal{P}^c, \mathcal{P}} Q_{\mathcal{P}}^{-1} Q_{\mathcal{P}, \mathcal{P}^c}] \right) \\
&= T(P_0) + \frac{s}{N} L_{\mathcal{P}^c} \odot \mathcal{T}_{\mathcal{P}}(Q)
\end{aligned}$$

This ends the proof of (52). ■

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